

Asset pricing in DSGE models — comparison of different approximation methods

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Abstract. There are many numerical methods suitable for approximating solutions of DSGE models. They differ in terms of accuracy, coding and computing time. However for many macroeconomic applications the differences in accuracy do not matter, since all methods generate approximations with similar statistical properties of simulated time series. In the paper we check whether this is also the case for DSGE models with financial variables, like stocks and risk-free bonds. These models are usually highly nonlinear and some special methods should be applied to approximate the asset prices. In the paper we take a simple macro-finance DSGE model proposed by Jermann, solve it with three different group of methods, simulate and check if the simulated series of financial variables differ in terms of basic statistical moments. For solving the model we use the higher-order perturbation approaches, the loglinear-lognormal method, as well as the Galerkin projection method. The results show that there might be significant differences between moments of the financial series in models approximated using different methods. For example for moderate parametrization the expected risk premium in the model approximated by the Galerkin method is about half of percentage point higher than for the perturbation methods and the loglinear lognormal approach. These results clearly indicate that the further research on the solution methods of DSGE models with financial variables is needed.

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JEL classification: C63, C68, G12

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1 Introduction

Dynamic stochastic general equilibrium models (DSGE) are one of the main tools used for analysis of economic policy. Having solid microfoundations makes them from the one hand robust to the Lucas critique but also very complicated. From the mathematical point of view a model is represented by a set of stochastic, nonlinear difference equations. There have been many methods proposed in literature for approximation of such systems [7]. They differ in terms of accuracy, speed and implementation difficulty [1, 5, 6]. The most popular are perturbation methods based on local polynomial approximation of a solution, projection methods seeking for a global approximation with Chebyshev polynomials and approaches based on solving Bellman's optimality principle. Despite significant differences in terms of accuracy, as far as macroeconomic variables are concerned time series simulated from models approximated with different methods usually have similar statistical properties. Therefore for many applications the simplest first-order approximations, like loglinearisation, provide sufficient accuracy. However this may not be the case if a macroeconomic model is extended to include asset prices as well, since to price stocks or bonds correctly it is crucial to have correct second- and higher order moments of payoffs and discount factors.

In this paper we study statistical properties of asset prices in a DSGE model approximated using several different methods. Contrary to previous papers by Aruoba, Fernandez-Villaverde and Rubio-Ramirez [5] and Heer and Maussner [6], we use a modification of otherwise standard stochastic growth model proposed by Jermann [8] that incorporates exogenous habits in a utility function and investment costs. These modifications enable the model to generate a significant risk premium and therefore are commonly included in more complex macroeconomic models. In contrast to previous studies, the model

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exhibits higher nonlinearities, that make it particularly hard to approximate accurately. We solve the model using several perturbation methods of different orders, the Galerkin variant of the projection approach as well as the loglinear-lognormal approach which is a method tailored to approximating asset price dynamics in DSGE models. We show that although statistical properties of macroeconomic series in the model are virtually the same across the methods, the behaviour of asset prices may differ significantly which indicates the need for the further research in that area.

The paper is organized as follows. In the first section we briefly introduce the model. Then we discuss the methods used for the approximation. Finally we present the results of the simulation study.

2 The model

The paper uses the model proposed by Jermann [8] with only one minor modification — we abstract from the long-run growth. The economy is populated by large number of identical households who evaluate the consumption stream according to the instantaneous utility function:

$$u(C_t, C_{t-1}) = \frac{(C_t - \chi_c C_{t-1})^{1-\nu} - 1}{1-\nu}, \quad (1)$$

where C_t represents consumption, ν is the household's relative risk aversion, and χ_c is a habit persistence parameter. The households receive the income from work $W_t L_t$, where W_t is wage and L_t represents the fraction of time devoted to work. Since the households own firms, they also receive dividends D_t . So the budget constraint of the representative household has the following form:

$$W_t L_t + D_t = C_t. \quad (2)$$

In every period the household maximizes its expected lifetime utility:

$$\max_{C_t} \mathbb{E}_t \left[\sum_{h=0}^{\infty} \beta^h u(C_{t+h}, C_{t+h-1}) \right] \quad \text{s.t.} \quad C_{t+h} = W_{t+h} L_{t+h} + D_{t+h}, \quad h = 0, 1, \dots \quad (3)$$

where β represents the household's discount factor.

The representative firm combines capital K_t with labour to produce single good Y_t according to a standard Cobb–Douglas technology:

$$Y_t = Z_t K_t^\alpha L_t^{1-\alpha}, \quad (4)$$

where α represents capital share in the output, whereas Z_t is a stochastic shock with AR(1) law of motion:

$$\ln Z_t = \rho \ln Z_{t-1} + \sigma \epsilon_t, \quad \epsilon_t \sim N(0, 1). \quad (5)$$

The capital stock owned by the firm depreciates at a constant rate δ per period and is increased by investment I_t , so its evolution is given by:

$$K_t = K_{t-1} - \delta K_{t-1} + \Phi \left(\frac{I_t}{K_{t-1}} \right) K_{t-1}; \quad \Phi \left(\frac{I_t}{K_{t-1}} \right) = \frac{a_1}{1 - 1/\xi} \left(\frac{I_t}{K_{t-1}} \right)^{1-1/\xi} + a_0. \quad (6)$$

Function Φ is a concave function capturing the idea that adjusting capital rapidly is more costly than changing it slowly. Each period the firm decides how much labour to hire and how much to invest trying to maximise utility of the dividend stream paid to the shareholders:

$$\max_{L_t, I_t, K_t} \mathbb{E}_t \left[\sum_{h=0}^{\infty} \beta^h MU_{t+h} D_{t+h} \right] \quad \text{s.t.} \quad K_{t+h} = \left[1 - \delta + \Phi \left(\frac{I_{t+h}}{K_{t+h-1}} \right) \right] K_{t+h-1}, \quad h = 0, 1, \dots, \quad (7)$$

where a marginal utility of the household MU_t evolves according to:

$$MU_t = (C_t - \chi_c C_{t-1})^{-\nu} - \chi_c \beta \mathbb{E}_t \left[(C_{t+1} - \chi_c C_t)^{-\nu} \right] \quad (8)$$

and D_t corresponds to a net profit of the firm:

$$D_t = Y_t - W_t L_t - I_t \quad (9)$$

Since labour do not enter the utility function and its marginal product is always positive the households choose:

$$L_t = 1. \quad (10)$$

Wage equals the marginal product of labour:

$$W_t = (1 - \alpha)Z_t K_{t-1}^\alpha. \quad (11)$$

The firm's first-order optimality conditions imply:

$$Q_t = \beta \mathbb{E}_t \left[\frac{MU_{t+1}}{MU_t} \left(\alpha Z_{t+1} K_t^{\alpha-1} - \frac{I_{t+1}}{K_t} + Q_{t+1} \left[1 - \delta + \Phi \left(\frac{I_{t+1}}{K_t} \right) \right] \right) \right], \quad (12)$$

where Q_t is a Lagrange multiplier associated with the capital law of motion constraint in the decision problem (7):

$$Q_t = \left[\Phi' \left(\frac{I_t}{K_{t-1}} \right) \right]^{-1}. \quad (13)$$

The model has 10 macroeconomic variables and consists of 10 equations: (2), (4)–(6), (8)–(13). It can also easily incorporate asset prices. For example, stock P_t and 1-period risk-free bond $P_{f,t}$ price dynamics are given by the standard formulas:

$$P_t = \beta \mathbb{E}_t \left[\frac{MU_{t+1}}{MU_t} (P_{t+1} + D_{t+1}) \right], \quad (14)$$

$$P_{f,t} = \beta \mathbb{E}_t \left[\frac{MU_{t+1}}{MU_t} \right]. \quad (15)$$

3 Approximation methods

The model introduced in the previous section can be compactly written as follows:

$$\mathbb{E}_t f(\mathbf{X}_{t+1}, \mathbf{X}_t, \mathbf{X}_{t-1}, \sigma \boldsymbol{\epsilon}_t) = \mathbf{0}, \quad \boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}), \quad (16)$$

where \mathbf{X}_t is a vector of all n_x model variables, $\boldsymbol{\epsilon}_t$ represents a vector of n_e stochastic shocks and $f: \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \rightarrow \mathbb{R}^{n_x}$. We look for a solution of the general form:

$$\mathbf{X}_t = g(\mathbf{X}_{t-1}, \sigma \boldsymbol{\epsilon}_t), \quad (17)$$

which can also be expressed in terms of a smaller set of state variables \mathbf{S}_t :

$$\mathbf{X}_t = g_x(\mathbf{S}_{t-1}, \sigma \boldsymbol{\epsilon}_t), \quad \mathbf{S}_t = g_s(\mathbf{S}_{t-1}, \sigma \boldsymbol{\epsilon}_t). \quad (18)$$

For the analysed model the set of the state variables consists of consumption C_{t-1} , the capital stock K_{t-1} and the productivity shock Z_t . In the next subsections we discuss three approaches used for approximating the solution (17).

3.1 Perturbation methods

These methods are based on a local approximation of the solution with Taylor's polynomials around the model's deterministic steady state. This steady state can be seen as a limit of a system without any shocks, it is when $\sigma = 0$. Here we discuss only linearisation, the simplest of the perturbation approaches.

If we linearise the solution (17) around the steady state $\bar{\mathbf{X}}$, we get:

$$\mathbf{X}_t \approx \bar{\mathbf{X}} + \mathbf{G}_x(\mathbf{X}_{t-1} - \bar{\mathbf{X}}) + \mathbf{G}_\epsilon \sigma \boldsymbol{\epsilon}_t, \quad (19)$$

where \mathbf{G}_x and \mathbf{G}_ϵ are Jacobians of the function g with respect to \mathbf{X}_{t-1} and $\boldsymbol{\epsilon}_t$ respectively. The values of \mathbf{G}_x and \mathbf{G}_ϵ can be calculated by inserting (17) into (16) and using the implicit function theorem, which states that since the l.h.s. of (16) is equal to 0, this must also be the case for all its derivatives. This

in turn leads to a matrix quadratic equation [6]. The linearised solution with the state variables has the following form:

$$\hat{\mathbf{S}}_t \approx \mathbf{M}\hat{\mathbf{S}}_{t-1} + \mathbf{W}\boldsymbol{\epsilon}_t, \quad \hat{\mathbf{X}}_t \approx \mathbf{M}_X\hat{\mathbf{S}}_{t-1} + \mathbf{W}_X\boldsymbol{\epsilon}_t, \quad (20)$$

where hats over the variables represent deviations from the steady state: $\hat{\mathbf{S}}_t = \mathbf{S}_t - \bar{\mathbf{S}}$ and the matrices \mathbf{M} , \mathbf{W} , \mathbf{M}_X and \mathbf{W}_X consist of the elements of \mathbf{G}_x and \mathbf{G}_ϵ .

The higher-order approximations can be obtained in similar, recursive way by utilizing results of the approximation of a lower order as shown by Schmitt-Grohé and Uribe [9]. The method discussed here is very popular since it can be easily applied to models with large number of state variables. It is also very fast. Moreover it has been implemented in Matlab's Dynare package [3]. The main weakness of the approach is lack of accuracy since the obtained solution is close to the true one only near the steady state. When the system is far from the long-run equilibrium the approximation may be poor and in extreme cases the approximation may be divergent. However the higher-order approximations are considered by some researchers [5] to provide the accuracy of the similar order to other, more reliable approaches.

3.2 Loglinear lognormal approach

This method was proposed by Jermann [8] exclusively for the approximation of the asset price dynamics in linearised models. Since the linearisation abstracts from any second-order effects it cannot be used in models with asset prices. The loglinear-lognormal approach utilizes loglinear solution for the macroeconomic variables and extends it to include second-order terms for approximating asset prices. The loglinear solution of the model has the form (20):

$$\hat{\mathbf{s}}_t \approx \mathbf{M}\hat{\mathbf{s}}_{t-1} + \mathbf{W}\boldsymbol{\epsilon}_t, \quad \hat{\mathbf{x}}_t \approx \mathbf{M}_X\hat{\mathbf{s}}_{t-1} + \mathbf{W}_X\boldsymbol{\epsilon}_t \quad (21)$$

where $\hat{\mathbf{s}}_t = \ln \mathbf{S}_t - \ln \bar{\mathbf{S}}$. From (15) we have:

$$\begin{aligned} P_{f,t} &= \beta \mathbb{E}_t \left[\frac{MU_{t+1}}{MU_t} \right] = \beta \mathbb{E}_t \exp \left[\hat{\lambda}_{t+1} - \hat{\lambda}_t \right] = \beta \mathbb{E}_t \exp \left[\mathbf{M}_\lambda \hat{\mathbf{s}}_t + \mathbf{W}_\lambda \boldsymbol{\epsilon}_{t+1} - \mathbf{M}_\lambda \hat{\mathbf{s}}_{t-1} - \mathbf{W}_\lambda \boldsymbol{\epsilon}_t \right] = \\ &= \beta \mathbb{E}_t \exp \left[\mathbf{M}_\lambda (\mathbf{M}\hat{\mathbf{s}}_{t-1} + \mathbf{W}\boldsymbol{\epsilon}_t) + \mathbf{W}_\lambda \boldsymbol{\epsilon}_{t+1} - \mathbf{M}_\lambda \hat{\mathbf{s}}_{t-1} - \mathbf{W}_\lambda \boldsymbol{\epsilon}_t \right] = \\ &= \beta \mathbb{E}_t \exp \left[(\mathbf{M}_\lambda \mathbf{M} - \mathbf{M}_\lambda) \hat{\mathbf{s}}_{t-1} + (\mathbf{W}_\lambda \mathbf{M} - \mathbf{W}_\lambda) \boldsymbol{\epsilon}_t + \mathbf{W}_\lambda \boldsymbol{\epsilon}_{t+1} \right] = \\ &= \beta \exp \left[(\mathbf{M}_\lambda \mathbf{M} - \mathbf{M}_\lambda) \hat{\mathbf{s}}_{t-1} + (\mathbf{W}_\lambda \mathbf{M} - \mathbf{W}_\lambda) \boldsymbol{\epsilon}_t + 0.5 \mathbf{W}'_\lambda \mathbf{W}_\lambda \right], \end{aligned} \quad (22)$$

where in derivation we used the facts that: $\hat{\lambda}_t = \ln MU_t - \ln \bar{M}U$, $\hat{\lambda}_{t+1}$ and $\hat{\lambda}_t$ follow (21) and the expected value of the lognormal random variable $\mathbb{E}[\exp(x)] = \exp[\mathbb{E}(x) + 0.5\mathbb{D}^2(x)]$. The same approach can be applied to pricing stocks using discounted dividend version of the pricing equation, as shown by Jermann [8] and Acedański [2].

Similar to the perturbation approaches the presented method is easy to implement and can be applied to models with a large number of the state variables. However little is known about accuracy of the solution. Moreover the loglinear-lognormal framework treats the macroeconomic variables and the financial variables in different ways which sometimes is also considered as a weakness.

3.3 Projection method

The projection methods approximate globally either some parts of the solution (17) or some parts of the system (16) using linear combinations of Chebyshev polynomials $T_m(x)$ ($T_0(x) = 1$, $T_1(x) = x$, $T_m(x) = 2xT_{m-1}(x) - T_{m-2}(x)$, m - polynomial order). Following Heer and Maussner [6] we use both: we look for the approximating function $\Psi^{(1)}$ for the solution for Q_t and for the approximation $\Psi^{(2)}$ of the conditional expectation in the marginal utility dynamics (8) of the form:

$$\Psi^{(n)}(C, K, Z, \boldsymbol{\psi}^{(n)}) = \sum_{i,j,l} \psi_{i,j,l}^{(n)} T_i(C) T_j(K) T_l(Z), \quad i + j + l = m, \quad n = 1, 2, \quad (23)$$

For simplification we omit the time subscripts in the approximating functions (23). The unknown coefficients $\psi_{i,j,l}^{(1)}$ and $\psi_{i,j,l}^{(2)}$ should be set so to make the approximation functions as close as possible to the true functions within a given space of (C, K, Z) . Then if we know the solution for Q_t we can easily find the conditional solution for all other variables. To find the values of $\psi_{i,j,l}^{(n)}$ distance measures $R^{(n)}$ need to

be defined. For the first equation $R^{(1)}(C, K, Z, \psi^{(1)})$ is the difference between $\Psi^{(1)}$ and the r.h.s of (12) conditional on $\psi^{(1)}$. For the second one $R^{(2)}(C, K, Z, \psi^{(2)})$ is the difference between $\Psi^{(2)}$ and the expectation in (8) calculated conditional on $\psi^{(2)}$. The expected values in both expressions are approximated using Gauss-Hermite quadrature formula with 10 nodes.

Then to find the values of $\psi_{i,j,l}^{(n)}$ we use the Galerkin condition: the coefficients should be set so to make the residuals orthogonal to the Chebyshev polynomials:

$$\int_{\underline{C}}^{\overline{C}} \int_{\underline{K}}^{\overline{K}} \int_{\underline{Z}}^{\overline{Z}} R^{(n)}(C, K, Z, \psi^{(n)}) \cdot T_i(C) \cdot T_j(K) \cdot T_l(Z) dZ dK dC = 0 \quad (24)$$

The quadratures above are approximated using Gauss-Chebyshev formula with 20 nodes in each dimension. For both polynomials we use order $m = 6$, therefore the whole system of equations has 168 unknowns.

The projection methods are considered to be the most accurate [7], even far from the steady state. However they are hard to implement and are very slow. They also suffer from the curse of dimensionality. To approximate accurately the model with only 3 state variables the system of equations with more than 150 unknowns must be solved.

4 Results of the simulation study

For the simulation study we utilize rather standard parametrization and set $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.0136$, $\nu = 5$, $\rho = 0.95$, $\sigma = 0.01$. Only for the habit strength $\chi_c = 0.7$ and the curvature of the investment function $\xi = 0.8$ we use the values that slightly differ from the literature (Jermann [8] uses $\chi_c = 0.82$ and $\xi = 0.23$, whereas Heer and Maussner [6] study the model with $\chi_c = 0.8$ and $\xi = 0.23$). That parametrization do not allow the model to match the observed expected risk premium of about 6% per annum, but it makes the model less nonlinear and therefore easier to approximate, especially for the Galerkin method. We compare the following methods: perturbations in logs of the second-, third- and fifth-order, the loglinear-lognormal approach and the Galerkin method. For the given parametrization we approximate the model with these methods and run 1000 simulations with 250 quarters each.

Table 1 contains the basic moments of the macroeconomic variables. Since for all perturbation methods the results are exactly the same, we report them in one joint row. The table clearly shows that there are no important differences between the methods as far as the macroeconomic variables are concerned.

Method	Moments							
	$\mathbb{D}(Y)$	$\frac{\mathbb{D}(C)}{\mathbb{D}(Y)}$	$\frac{\mathbb{D}(I)}{\mathbb{D}(Y)}$	ar(Y)	ar(C)	ar(I)	corr(C, Y)	corr(I, Y)
loglinear lognormal perturbations	0.013	0.52	3.25	0.71	0.90	0.57	0.87	0.95
Galerkin	0.013	0.57	3.19	0.71	0.90	0.55	0.87	0.95

HP-filtered quarterly averages over 1000 simulations of 250 quarters; ar – autocorrelation coefficient; corr – correlation coefficient.

Table 1 Moments of the simulated macroeconomic variables

In table 2 we report the moments of the financial variables. Two observations are worth noting. First, as far as the perturbations and loglinear-lognormal approach are concerned the differences between the moments are negligible but still at least of one order of magnitude higher than in case of the macroeconomic variables. And second, there are significant differences between the moments for the Galerkin approach and the rest of the methods, especially in case of the expected risk premium. For the former the premium is about 1.4 percentage point, whereas for the latter it is less than 1 percentage point. So there is more than 40% difference, which for more extreme calibrations can be much higher in absolute values. The nonnegligible differences are also observed for the standard deviation of the risk-free rate (5.4 p.p. – 4.7 p.p.), the standard deviation of the dividend growth rate (2.9 p.p – 3.15 p.p) and the expected dividend/price ratio (4.3 – 4.0). These results are in sharp contrast with Jermann [8] who found no differences between the projection method and loglinear-lognormal approach in his model, but are supported by the results of Aldrich and Kung [4], who also obtained significant differences in asset price moments for the projection method and the standard perturbation techniques.

Method	Moments								
	$\mathbb{E}(R)$	$\mathbb{E}(R_f)$	$\mathbb{E}(R - R_f)$	$\mathbb{D}(R)$	$\mathbb{D}(R_f)$	$\mathbb{D}(\Delta D)$	$\mathbb{E}(DP)$	$\mathbb{D}(DP)$	$\text{corr}(R, R_f)$
loglin-lognor	4.44	3.54	0.90	8.34	4.70	3.12	4.04	0.50	-0.50
perturb. 2	4.39	3.43	0.96	8.35	4.65	3.16	3.99	0.49	-0.50
perturb. 3	4.43	3.57	0.86	8.34	4.71	3.17	4.02	0.50	-0.50
perturb. 5	4.41	3.47	0.94	8.34	4.68	3.17	4.00	0.49	-0.50
Galerkin	4.71	3.31	1.40	8.27	5.40	2.88	4.32	0.52	-0.49

Annualized averages in p.p. over 1000 simulations of 250 quarters; \mathbb{E} – unconditional mean, \mathbb{D} – unconditional standard deviation, corr – correlation coefficient, R – stock return, R_f – risk-free rate, ΔD – dividend growth rate, DP – dividend/price ratio.

Table 2 Moments of the simulated financial variables

5 Conclusion

In the paper using the model proposed by Jermann we have shown that despite the differences in accuracy all the approximation methods generate virtually the same moments of the main macroeconomic variables. However for the financial variables the differences between the perturbation method and the projection approaches are much higher. For moderate parametrization the expected risk premium in the model approximated by the Galerkin method is about 0.5 percentage point higher than for the perturbation methods and the loglinear lognormal approach. But it must be made clear that although the projection methods are considered to be the most accurate we cannot find out which method gives the moments that are closer to the true values in that particular case. Nonetheless these results clearly indicate that the further research on the solution methods of DSGE models with financial variables is needed.

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