Application of Cooperative Game Solution Concepts to a Collusive Oligopoly Game

David Bartl[1]

Abstract. An oligopoly is a market where a couple of large producers supply some goods. If the oligopoly is collusive, the producers form coalitions. Then, within each of the coalitions, the producers wish to divide their total profit among themselves. They could use a cooperative transferable utility game solution concept, such as the core, if the game were in the coalitional form. This, however, is not the case. In this paper, we propose an approach to overcome that difficulty: converting the collusive oligopoly into the partition function form, we show how the known cooperative game solution concepts (core, bargaining set) can be applied to that game. Actually, the proposed approach is suitable not only for a collusive oligopoly, but, under some assumptions, for any cooperative strategic form game.

Keywords: cooperative games, partition function form games, solution concepts, core, bargaining set.

JEL classification: C71, L13
AMS classification: 91A12

1 Introduction

Let us consider a game in the *strategic* (or *normal*) *form*: let $N = \{1, \dots, n\}$ be the set of the *players*, let $X_1, \dots, X_n$ be the *strategy spaces* of the players, and let $F_1, \dots, F_n$ be their *payoff functions*. Each X_j is a non-empty set and each F_j is a real function defined on the Cartesian product $X_1 \times \cdots \times X_n$.

For a player $j \in N$, the set X_j consists of all the decisions (called *strategies*) which the player j can make. At a moment, each of the players $j \in N$ picks up a strategy $x_j \in X_j$ and receives the amount of $F_j(x_1, \dots, x_n)$ of some utility (such as money).

An example of a game in the strategic form is an *oligopoly*. It is a market where a couple of large producers supply some goods. Each of the producers has enough power to influence the market by its decisions. We shall describe the *Cournot model* [3] of an oligopoly here. Let $N = \{1, \dots, n\}$ be the set of the oligopolists. They supply one kind of some goods, product or commodity (such as metal, grain, oil, etc.). For $j \in N$, let $L_j > 0$ be the *production limit* of the oligopolist j, i.e. the maximum amount of the goods the oligopolist is able to supply to the market. Then $X_j = \langle 0, L_j \rangle$, a closed interval, is the oligopolist's strategy space. Now, each of the oligopolists decides to supply some amount $x_j \in X_j$ of the goods to the market. Hence, the total supply of the goods is $s = \sum_{j=1}^{n} x_j$. Then an internal mechanism of the market, which effects so that the market clears (the supply equals the demand for the goods), establishes the price $p(s)$ per a unit of the goods. The function p is the *price* (or *inverse demand*) *function* of the oligopoly. The price function p is a real function defined on the closed interval $\langle 0, L \rangle$ where $L = \sum_{j=1}^{n} L_j$ is the maximum total supply of the goods to the market. Then $x_j p(s)$, the supply multiplied by the unit price, is the revenue of the oligopolist j. However, the oligopolist faces some production costs connected with the supply of the amount x_j of the goods. The oligopolist's production costs are $c_j(x_j)$ where c_j is the *cost function* of the oligopolist j. The cost function c_j is a real function defined on the interval $X_j = \langle 0, L_j \rangle$. Finally, the oligopolist's net profit is $F_j(x_1, \dots, x_n) = x_j p(s) - c_j(x_j)$ for $j \in N$ where $s = \sum_{j=1}^{n} x_j$.

Given a strategic form game (oligopoly), we say that $[x_1^*, \dots, x_n^*] \in X_1 \times \cdots \times X_n$ is a point of the *Nash equilibrium* iff, for each $j \in N$, it holds $F_j(x_1^*, \dots, x_{j-1}^*, x_j, x_{j+1}^*, \dots, x_n^*) \leq F_j(x_1^*, \dots, x_{j-1}^*,$

[1]University of Ostrava, Faculty of Science, Department of Mathematics, 30. dubna 22, 701 03 Ostrava, Czech Republic, e-mail: bartl@osu.cz

$x_j^*, x_{j+1}^*, \ldots, x_n^*)$ for all $x_j \in X_j$. The Nash equilibrium is also termed the *Cournot equilibrium* or the *Cournot-Nash equilibrium* if the strategic form game under consideration is a Cournot oligopoly.

Let us assume the given strategic form game (oligopoly) is non-cooperative in the following sense: (A) each player j chooses the decision $x_j \in X_j$ independently of the other players and, at the same time, maximizes own profit $F_j(x_1, \ldots, x_n)$, disregarding the payoffs of the remaining players; (B) if the situation has "settled down", i.e., is in a Nash equilibrium, then the players do not react at all (or react very slowly) if only one of them tries to change the equilibrium state; as the player's payoff F_j does not increase, the player who tried to change the situation is returned back to the equilibrium state. Provided these two assumptions (A) and (B) are satisfied and the game has exactly one point of the Nash equilibrium, let us assume that the situation arrives there. Then, if we manage to find the unique point $[x_1^*, \ldots, x_n^*]$ of the Nash equilibrium, we shall also know the individual payoffs $F_j(x_1^*, \ldots, x_n^*)$, which the players will receive.

The question whether a point of the Nash equilibrium exists is addressed by the original papers of Nash [11, 12], by the classical paper of Nikaidô and Isoda [13], or by more recent papers, e.g. [14]. Conditions for the existence and uniqueness of the Cournot equilibrium can be found, e.g., in the papers [8, 5]. (See also [4].) Especially interesting are the papers of Szidarovszky and Yakowitz [16, 17] and the book [15].

Now, let us consider that the given strategic form game is cooperative (the given oligopoly is collusive). That is, the players form coalitions. A *coalition* is any subset of the set of the players N. Let $S = \{j_1, \ldots, j_{n_S}\} \subseteq N$ be a coalition which has emerged. The members $j_1, \ldots, j_{n_S}$ of the coalition coordinate their decisions in order to maximize their total payoff. Naturally, the coalition's strategy space X_S comprises the Cartesian product $X_{j_1} \times \cdots \times X_{j_{n_S}}$; when the players choose a strategy from the Cartesian product, the total payoff F_S of the coalition is the sum $\sum_{j \in S} F_j$ of the individual payoffs of its members. However, the strategy space X_S may contain yet additional strategies, which are not in the Cartesian product, i.e., they are feasible only if the players join because a single player or a smaller group of the players may not have enough power to realise those decisions.

The situation simplifies if the considered cooperative game is a collusive Cournot oligopoly. Given a coalition $S = \{j_1, \ldots, j_{n_S}\} \subseteq N$, the oligopolists $j_1, \ldots, j_{n_S}$ simply join their production capacities so that the coalition's strategy space X_S reduces to the closed interval $\langle 0, L_S \rangle$ where $L_S = \sum_{j \in S} L_j$. When the coalition decides to supply some amount $x_S \in X_S$ of the goods to the market, the oligopolists $j_1, \ldots, j_{n_S}$ will allocate their production to their most efficient plants. Hence, the coalition's cost function c_S is calculated as $c_S(x_S) = \min\{ \sum_{j \in S} c_j(x_j) : \sum_{j \in S} x_j = x_S \text{ with } x_j \in X_j \text{ for } j \in S \}$, provided that the minimum exists. Therefore, the coalition's net profit is $F_S = x_S p(s) - c_S(x_S)$ where s is the total supply of the goods to the market.

It is definitely beyond the scope of this paper to study the process of the formation of coalitions. Following [10], we shall simply assume that a coalition structure will "crystallize". A *coalition structure* is any partition of the set N. In other words, a coalition structure is a collection $\mathcal{S} = \{S_1, \ldots, S_\nu\}$ of coalitions such that $\bigcup_{\iota=1}^{\nu} S_\iota = N$ and $S_{\iota'} \cap S_{\iota''} = \emptyset$ iff $\iota' \neq \iota''$ for all $\iota', \iota'' = 1, \ldots, \nu$.

Thus, we obtain another strategic form game where the set of the players is the set of the established coalitions $\mathcal{S} = \{S_1, \ldots, S_\nu\}$, their strategy spaces are $X_{S_1}, \ldots, X_{S_\nu}$ and their payoff functions are $F_{S_1}, \ldots, F_{S_\nu}$. It is quite natural to assume that, in this game, the coalitions will behave in a mutually non-cooperative way, making the above assumptions (A) and (B) hold. Hence, if there exists exactly one point $[x_{S_1}^*, \ldots, x_{S_\nu}^*]$ of the Nash equilibrium in the game, the total payoffs $F_{S_\iota}(x_{S_1}^*, \ldots, x_{S_\nu}^*)$ of the coalitions can be determined for $\iota = 1, \ldots, \nu$.

Now, let the members $j_{\iota,1}, \ldots, j_{\iota,n_\iota}$ of an established coalition $S_\iota = \{j_{\iota,1}, \ldots, j_{\iota,n_\iota}\} \in \mathcal{S}$ wish to divide their total profit F_{S_ι} among themselves. If the game were in the coalitional form (see Section 3), then the members could use, e.g., the concept of the core [7], the bargaining set [10], or another transferable utility (TU) game solution concept. That, however, is not the case: the cooperative game under consideration is in the strategic form.

In this paper, in order to apply TU-game solution concepts to the given cooperative strategic form game, we propose the following approach: First, to convert the cooperative strategic form game into the partition function form (Section 2). And, then, to apply cooperative TU-game solution concepts to the partition function form game (Section 3). The approach proposed here extends and generalises the author's earlier idea which originally appeared in [1].

2 Conversion of a cooperative strategic form game into a partition function form game

Let $N = \{1, \dots, n\}$ be the set of the players. Let $\mathfrak{S}$ be the set of all coalition structures $\mathcal{S}$ that the players can form. Recall that a *partition function form game*, in the famous sense of Thrall and Lucas [18], is given by a *partition function* $\mathcal{F}$ which is defined on the set of all coalition structures $\mathfrak{S}$ and, to each coalition structure $\mathcal{S} \in \mathfrak{S}$, it assigns a function $\mathcal{F}_{\mathcal{S}} \colon \mathcal{S} \to \mathbb{R}$. For a coalition $S \in \mathcal{S}$, the value $\mathcal{F}_{\mathcal{S}}(S)$ is the total payoff that the established coalition S will receive.

Given a cooperative game in the strategic form, it is easy to convert it into the partition function form: Let $\mathcal{S} = \{S_1, \dots, S_\nu\} \in \mathfrak{S}$ be a coalition structure, which the players have formed. Let $X_{S_1}, \dots, X_{S_\nu}$ and $F_{S_1}, \dots, F_{S_\nu}$ be the strategy spaces and the payoff functions, respectively, of the coalitions. Let us assume that, *for any coalition structure $\mathcal{S} \in \mathfrak{S}$, there exists exactly one Nash equilibrium in the game among the coalitions.* Let $[x^*_{S_1}, \dots, x^*_{S_\nu}] \in X_{S_1} \times \cdots \times X_{S_\nu}$ be the unique point of the Nash equilibrium. We put $\mathcal{F}_{\mathcal{S}}(S_\iota) = F_{S_\iota}(x^*_{S_1}, \dots, x^*_{S_\nu})$ for $\iota = 1, \dots, \nu$. The conversion has been described thus.

The assumption of the existence and uniqueness of the Nash equilibrium for any coalition structure $\mathcal{S} \in \mathfrak{S}$ can be seen quite restrictive. However, it can be met in the case of a Cournot oligopoly for example. It is not difficult to show [16, 17, 15] that if the price function p of the oligopoly is linear and decreasing (so $p(s) = as + b$ for some $a < 0$ and $b > 0$) and the oligopolists' cost functions c_j are non-increasing and convex, then the oligopoly possesses exactly one Cournot equilibrium. Moreover, it is an exercise to show that if the cost functions c_j are convex, then, for any coalition $S \subseteq N$, the coalition's cost function $c_S(x_S) = \min\bigl\{ \sum_{j \in S} c_j(x_j) : \sum_{j \in S} x_j = x_S \text{ with } x_j \in X_j \text{ for } j \in S \bigr\}$ is also convex. Hence, there exists a unique Cournot equilibrium for any coalition structure $\mathcal{S} \in \mathfrak{S}$ in the non-cooperative oligopolistic game among the coalitions.

3 Application of Cooperative TU-Game Solution Concepts to a Partition Function Form Game

Let us consider a cooperative game with transferable utility (TU) in the *coalitional form*: Let $N = \{1, \dots, n\}$ be the set of the players. A *coalition* is any subset of the set of the players. Then $\mathcal{P}(N) = \{ S : S \subseteq N \}$, the potency set of the set N, is the collection of all coalitions which can be formed. Finally, let $v \colon \mathcal{P}(N) \to \mathbb{R}$ with $v(\emptyset) = 0$ be the *coalitional* (or *characteristic*) *function* of the game.

When a coalition $S \subseteq N$ is formed, it receives the amount of $v(S)$ units of some utility. It is assumed here that the utility is *transferable*, that is, the members of the coalition S can divide the amount among themselves.

Let us consider that the players have formed a coalition structure $\mathcal{S} = \{S_1, \dots, S_\nu\}$. Then $v(S_\iota)$ is the payoff that the coalition S_ι receives for $\iota = 1, \dots, \nu$. Now, the question, which the cooperative game theory studies, is how will the members of the coalitions S_ι divide their payoffs $v(S_\iota)$ among themselves.

The division of the profit among the players is described by the payoff vector. A *payoff vector* is any n-tuple $a = [a_1, \dots, a_n] \in \mathbb{R}^n$. The number a_j stands for the amount which is allotted to the player $j \in N$.

Several solution concepts – such as the core [7] or the bargaining set [10] – were proposed to address the question. Recall that a *solution concept* is a mapping that, to a given coalitional function $v \colon \mathcal{P}(N) \to \mathbb{R}$ with $v(\emptyset) = 0$ and a given coalition structure $\mathcal{S}$, assigns a set of payoff vectors; sometimes, it assigns a collection of sets of payoff vectors or just a single payoff vector (in the case of the von Neumann-Morgenstern solution or the Shapley value, respectively; we shall not deal with these solution concepts in this paper).

Here, given a coalition structure $\mathcal{S}$, we would like to apply those solution concepts to a partition function form game $\mathcal{F}$. In the following, we shall recall and contemplate the solution concept of the core and that of the bargaining set.

Given a coalitional function $v \colon \mathcal{P}(N) \to \mathbb{R}$ with $v(\emptyset) = 0$ and a coalition structure $\mathcal{S} = \{S_1, \dots, S_\nu\}$, the *core* of the game is the set $\mathcal{C} = \bigl\{ a \in \mathbb{R}^n : \sum_{j \in S_\iota} a_j = v(S_\iota) \text{ for } S_\iota \in \mathcal{S} \text{ and } \sum_{j \in S} a_j \geq v(S) \text{ for all } S \in \mathcal{P}(N) \setminus \mathcal{S} \bigr\}$.

Now, given the coalition structure $\mathcal{S} = \{S_1, \ldots, S_\nu\}$, as above, and recalling the motivation stated in the Introduction, let us apply the concept of the core to a partition function form game $\mathcal{F}$. We can indeed formulate the equalities that are a part of the description of the core: let $\sum_{j \in S_\iota} a_j = \mathcal{F}_\mathcal{S}(S_\iota)$ for $S_\iota \in \mathcal{S}$. The equalities mean that each of the established coalitions S_ι divides all its profit $v(S_\iota)$ among its members. Nonetheless, how about the inequalities $\sum_{j \in S} a_j \geq v(S)$ for $S \in \mathcal{P}(N) \setminus \mathcal{S}$? Do we need the inequalities – what do they mean?

The inequalities $\sum_{j \in S} a_j \geq v(S)$ for $S \in \mathcal{P}(N) \setminus \mathcal{S}$ are the *conditions of group stability.* Let us consider a coalition $S \in \mathcal{P}(N) \setminus \mathcal{S}$. That is, the coalition does actually *not* exist, but *could* potentially be formed. Should the respective inequality not hold, so we would have $\sum_{j \in S} a_j < v(S)$, then the coalition S would have a good reason to form because its total payoff $v(S)$ will be higher that the present total payoff $\sum_{j \in S} a_j$ of its members. That is, if $\sum_{j \in S} a_j < v(S)$, then the present coalition structure $\mathcal{S}$ is instable and the new coalition S will emerge.

Let us continue that thoughts: When the new coalition $S \in \mathcal{P}(N) \setminus \mathcal{S}$ emerges, what happens with the coalition structure $\mathcal{S}$? We assume that a new coalition structure $\mathcal{S}_S$, containing S, will form shortly after the emergence of the coalition S. Which particular coalition structure $\mathcal{S}_S \in \mathfrak{S}$ will form, i.e., which coalitions it will contain, depends on the chosen approach. In this paper, we mention the γ-approach and the δ-approach of Hart and Kurz [6].

If we assume the γ-approach, then the new coalition structure will be $\mathcal{S}_S = \{S\} \cup \{\, S_\iota \in \mathcal{S} : S_\iota \cap S = \emptyset \,\} \cup \{\, \{j\} : \exists S_\iota \in \mathcal{S} \colon j \in S_\iota \setminus S \,\}$. In words, the new coalition structure $\mathcal{S}_S$ contains the new coalition S, all the formerly established coalitions $S_\iota \in \mathcal{S}$ not affected by the departure ($S_\iota \cap S = \emptyset$), but the other coalitions ($S_\iota \cap S \neq \emptyset$) split into singletons $\{j\}$.

If we assume the δ-approach, then the new coalition structure will be $\mathcal{S}_S = \{S\} \cup \{\, S_\iota \setminus S : S \not\supseteq S_\iota \in \mathcal{S} \,\}$. In words, the new coalition structure $\mathcal{S}_S$ contains the new coalition S, all the formerly established coalitions $S_\iota \in \mathcal{S}$ not affected by the departure ($S_\iota \cap S = \emptyset$), but the remaining non-empty parts $S_\iota \setminus S$ of the other coalitions ($S_\iota \cap S \neq \emptyset$) stay intact.

Now, it is easy to formulate the conditions of group stability for the partition function form game $\mathcal{F}$ and the established coalition structure $\mathcal{S}$. We write $\sum_{j \in S} a_j \geq \mathcal{F}_{\mathcal{S}_S}(S)$ for $S \in \mathcal{P}(N) \setminus \mathcal{S}$. To conclude, we define the *core* of the partition function form game $\mathcal{F}$ with respect to the coalition structure $\mathcal{S} = \{S_1, \ldots, S_\nu\}$ to be the set $\mathcal{C} = \{\, a \in \mathbb{R}^n : \sum_{j \in S_\iota} a_j = \mathcal{F}_\mathcal{S}(S_\iota)$ for $S_\iota \in \mathcal{S}$ and $\sum_{j \in S} a_j \geq \mathcal{F}_{\mathcal{S}_S}(S)$ for all $S \in \mathcal{P}(N) \setminus \mathcal{S} \,\}$.

Note that, in the definition of the core, the payoffs $\mathcal{F}_{\mathcal{S}_S}(S')$ of the other coalitions $S' \in \mathcal{S}_S \setminus \{S\}$ from the new coalition structure $\mathcal{S}_S$ are immaterial to us. However, when a coalition $S \in \mathcal{P}(N) \setminus \mathcal{S}$ departs, neither the original definition of the core of a coalitional form game considers what happens with the payoffs of the other coalitions.

We shall deal with the concept of the bargaining set in the rest of this section. We shall recall the concept of the imputation, objection, and counterobjection first.

Let a coalitional function $v \colon \mathcal{P}(N) \to \mathbb{R}$ with $v(\emptyset) = 0$ and a coalition structure $\mathcal{S} = \{S_1, \ldots, S_\nu\}$ be given. Then the set of the *imputations* of the game is the set $X = \{\, a \in \mathbb{R}^n : \sum_{j \in S_\iota} a_j = v(S_\iota)$ for $S_\iota \in \mathcal{S}$ and $a_j \geq v(\{j\})$ for all $j \in N \,\}$. As above, the equalities $\sum_{j \in S_\iota} a_j = v(S_\iota)$ mean that each of the established coalitions S_ι divides all its profit $v(S_\iota)$ among its members. The inequalities $a_j \geq v(\{j\})$ are the *conditions of individual rationality.* For a $j \in N$, the one-player coalition $\{j\}$ does actually not exist (unless $\{j\} \in \mathcal{S}$), but, if $a_j < v(\{j\})$, i.e., the player j receives less than the player can obtain by forming own independent coalition, then the present coalition structure $\mathcal{S}$ is instable and the coalition $\{j\}$ will emerge.

Consider two distinct players $k, l \in S_\iota \in \mathcal{S}$, $k \neq l$, from an established coalition S_ι. Let $a \in X$ be an imputation under the consideration.

An *objection* of the player k against l at the imputation a is a pair (K, b) where $K \subseteq N$ is a coalition such that $k \in K \not\ni l$ and $b \in \mathbb{R}^K$ is such that $\sum_{j \in K} b_j = v(K)$ and $b_j > a_j$ for all $j \in K$. That is, the coalition K does actually not exist, but has the potential to form because all its new members will receive higher payoffs than under the current division a of the profit. Note that the concept of the objection does not concern with the payoffs of the players outside the coalition K if the coalition separates.

A *counterobjection* of the player l to the objection (K, b) of k against l at a is a pair (L, c) where $L \subseteq N$ is a coalition such that $l \in L \not\ni k$ and $l \in \mathbb{R}^L$ satisfies $\sum_{j \in L} c_j = v(L)$ with $c_j \geq b_j$ for all $j \in L \cap K$ and $c_j \geq a_j$ for all $j \in L \setminus K$. So, it is assumed now that the coalition K has really emerged.

The coalition L does not actually exist, but again has the potential to form because the members from K as well as the new members from outside K will receive the same or higher payoffs than under the current division b or a, respectively, of the profit. Note that neither the concept of the counterobjection concerns with the payoffs of the players outside the coalition L if it forms.

We say that an objection is *justified* iff there is no counterobjection to it. Finally, the *bargaining set* is the set of all imputations $a \in X$ such that there does not exist any justified objection at a. That is, the bargaining set is the set $\mathcal{M}_1^i = \{\, a \in X : \forall S_\iota \in \mathcal{S}\ \forall k, l \in S_\iota,\ k \neq l,\ \forall (K,b),\ (K,b)$ is an objection of k against l at a, $\exists (L,c)$, (L,c) is a counterobjection of l to (K,b) of k against l at $a\,\}$.

Now, having understood the concept of the core earlier, it is easy to restate the above definitions in the setting of a partition function form game $\mathcal{F}$. Let $\mathcal{S} = \{S_1, \ldots, S_\nu\}$ be the established coalition structure. Recall that $\mathcal{S}_S$ denotes the coalition that will form if the coalition $S \in \mathcal{P}(N) \setminus \mathcal{S}$ decides to depart. We can adopt several approaches (such as the γ-approach or the δ-approach) to define $\mathcal{S}_S$. (We put $\mathcal{S}_S = \mathcal{S}$ if $S \in \mathcal{S}$.)

We define the set of the *imputations* to be the set $X = \big\{\, a \in \mathbb{R}^n : \sum_{j \in S_\iota} a_j = \mathcal{F}_{\mathcal{S}}(S_\iota)$ for $S_\iota \in \mathcal{S}$ and $a_j \geq \mathcal{F}_{\mathcal{S}_{\{j\}}}(\{j\})$ for all $j \in N \,\big\}$.

Let $k, l \in S_\iota \in \mathcal{S}$, $k \neq l$, be two distinct players from an established coalition and let $a \in X$ be an imputation. We define an *objection* of the player k against l at the imputation a to be a pair (K,b) where $K \subseteq N$ is such that $k \in K \not\ni l$ and $b \in \mathbb{R}^K$ is such that $\sum_{j \in K} b_j = \mathcal{F}_{\mathcal{S}_K}(K)$ and $b_j > a_j$ for all $j \in K$. And we define a *counterobjection* of the player l to the objection (K,b) of k against l at a to be a pair (L,c) where $L \subseteq N$ is such that $l \in L \not\ni k$ and $c \in \mathbb{R}^L$ satisfies $\sum_{j \in L} c_j = \mathcal{F}_{(\mathcal{S}_K)_L}(L)$ with $c_j \geq b_j$ for all $j \in L \cap K$ and $c_j \geq a_j$ for all $j \in L \setminus K$.

Finally, we define the *bargaining set* to be the set of all imputations $a \in X$ such that there does not exist any justified objection at a, i.e., to be the set $\mathcal{M}_1^i = \{\, a \in X : \forall S_\iota \in \mathcal{S}\ \forall k, l \in S_\iota,\ k \neq l,\ \forall (K,b)$, (K,b) is an objection of k against l at a, $\exists (L,c)$, (L,c) is a counterobjection of l to (K,b) of k against l at $a\,\}$.

4 Conclusions

We considered a cooperative game in the strategic form. The classical solution concepts (the core, the bargaining set, etc.), being defined for coalitional form games, cannot be applied to that game directly. Therefore, under the assumption of the existence and uniqueness of the Nash equilibrium, we proposed in Section 2 to convert the cooperative strategic form into a partition function form game.

We noted in Section 2 that if the price function of a Cournot oligopoly is linear and decreasing and the cost functions of the oligopolists are convex and non-increasing, then there exists exactly one Cournot equilibrium, whence the proposed conversion is possible. (See also [9].) It is a motivation of further research to find more general conditions under which there exists (exactly one) Cournot equilibrium in the oligopoly.

In Section 3, we showed how to apply the concept of the core and that of the bargaining set, which are defined for coalitional form games, to a partition function form game. Consequently, they can be applied to the original cooperative strategic form game (under the assumption of the existence and uniqueness of the Nash equilibrium), such as the collusive Cournot oligopoly.

For the lack of the space, we did not deal with other popular solution concepts (the von Neumann-Morgenstern solution, the kernel, the nucleolus, or the Shapley value) in Section 3. They could be applied analogously.

Note that if the considered partition function form game $\mathcal{F}$ is the result of the conversion of a cooperative strategic form game, if $\mathcal{S} = \{N\}$, i.e., the coalition structure contains only the grand coalition of the players, and if we assume the γ-approach, then our definition of the core of the partition function form game yields precisely the concept of the γ-core of Chander and Tulkens [2]. Nonetheless, our approach is more general in the sense that we define the core for *any* coalition structure $\mathcal{S}$.

Actually, it was essential for the conversion described in Section 3 to decide upon the approach which coalition structure $\mathcal{S}_S$ will form when a coalition $S \in \mathcal{P}(N) \setminus \mathcal{S}$ departs from $\mathcal{S}$. The application of the γ-approach, the δ-approach, etc., results in the concept of the γ-core, the δ-core, etc., the concept of the

γ-bargaining set, the δ-bargaining set, etc., etc. While our concept of the γ-core is more general than that of [2], as already mentioned, the concept of the γ-bargaining set or the δ-bargaining set is, according to the author's best knowledge, new.

References

[1] Bartl, D. Translation of a Collusive Oligopoly Game to a Coalitional Form Game. In: *Proceedings of the 19th International Conference Mathematical Methods in Economics 2001*, Vysoká škola ekonomická v Praze, 2001, 1–6. ISBN 80-245-0196-1.

[2] Chander, P., and Tulkens, H.: The Core of an Economy with Multilateral Environmental Externalities, *International Journal of Game Theory* **26** (1997), 379–401.

[3] Cournot, A. A.: *Recherches sur les principes mathématiques de la théorie des richesses.* L. Hachette, Paris, 1838. English translation by N. T. Bacon: *Researches into the Mathematical Principles of the Theory of Wealth.* Macmillan, New York, 1897, 1927. Reprinted: Augustus M. Kelly, New York, 1960, 1964, 1971. ISBN 0-678-00066-2.

[4] Everhart, C.: Cournot Oligopoly and Concavo-Concave Demand, *University of Zurich. Department of Economics: Working Paper Series*, Working Paper No. **16** (April 2011).

[5] Gaudet, G., and Salant, S. W.: Uniqueness of Cournot Equilibrium: New Results from Old Methods, *Review of Economic Studies* **58** (1991), 399–404.

[6] Hart, S., and Kurz, M.: Endogenous Formation of Coalitions, *Econometrica* **51** (1983), 1047–1064.

[7] Kannai, Y.: The Core and Balancedness. In: *Handbook of Game Theory with Economic Applications. Volume 1* (R. J. Aumann, and S. Hart, eds.), North-Holland, Amsterdam, 1992, 355–395. ISBN 0-444-88098-4.

[8] Kolstad, C. D., and Mathiesen, L.: Necessary and Sufficient Conditions for Uniqueness of a Cournot Equilibrium, *Review of Economic Studies* **LIV** (1987), 681–690.

[9] Lardon, A.: The γ-core in Cournot oligopoly TU-games with capacity constraints, *Theory and Decision* **72** (2011), 387–411.

[10] Maschler, M.: The Bargaining Set, Kernel, and Nucleolus. In: *Handbook of Game Theory with Economic Applications. Volume 1* (R. J. Aumann, and S. Hart, eds.), North-Holland, Amsterdam, 1992, 591–667. ISBN 0-444-88098-4.

[11] Nash, J., Jr.: Equilibrium Points in N-Person Games, *Proceedings of the National Academy of Sciences of the United States of America* **36** (1950), 48–49.

[12] Nash, J.: Non-Cooperative Games, *Annals of Mathematics* **54** (1951), 286–295.

[13] Nikaidô, H., and Isoda, K.: Note on Noncooperative Convex Games, *Pacific Journal of Mathematics* **5** (1955), 807–815.

[14] Nishimura, K., and Friedman, J.: Existence of Nash Equilibrium in n Person Games without Quasi-Concavity, *International Economic Review* **22** (1981), 637–648.

[15] Okuguchi, K., and Szidarovszky, F.: *The Theory of Oligopoly with Multi-Product Firms.* Springer-Verlag, Berlin, 1990. ISBN 3-540-52567-X.

[16] Szidarovszky, F., and Yakowitz, S.: A New Proof of the Existence and Uniqueness of the Cournot Equilibrium, *International Economic Review* **18** (1977), 787–789.

[17] Szidarovszky, F., and Yakowitz, S.: Contributions to Cournot Oligopoly Theory, *Journal of Economic Theory* **28** (1982), 51–70.

[18] Thrall, R. M., and Lucas, W. F.: n-Person Games in Partition Function Form, *Naval Research Logistics Quarterly* **10** (1963), 281–298.