Resolved and unresolved problems in the theory of redistribution systems

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Abstract. This study adopts the approach (and context) taken by J. Neumann and O. Morgenstern in describing, defining and finding solutions to simple majority game of three players and applies it to finding similar solutions in the redistribution system of three players. This system allows us to analyse situations in which the volume of what can be divided between players is determined by the way the players divide it, i.e. it is one of the examples of a non-constant sum game. As we anticipate a fully symmetric situation, we may define the term of "expected average payoff". From the term "expected average payoff" the concept of commonly acceptable equilibrium is derived. The distribution of wage at an acceptable equilibrium point is (in general) close to Nash's solution to a relevant cooperative game, which is derived from the point whose coordinates coincide with the expected average wage, yet are not completely identical. In the conclusion we outline the practical use of the model based on the definition of the redistribution system as well as a commonly acceptable equilibrium in the context of testing hypotheses concerning the objects called structures based on mutual covering-up of violation of generally accepted principles. The concept of these structures was derived from analysing games such as The Tragedy of the Commons, in the context of current problems (e.g. corruption and related issues).

Keywords: simple majority game, redistribution system, discrimination equilibrium, commonly acceptable equilibrium, structures based on mutual covering-up.

JEL Classification: C70
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1 Three-person games in the classic book theory of games and economic behavior

Published for the first time as long ago as in 1944, the classic work by J. Neumann and O. Morgenstern Theory of Games and Economic Behavior [8] provides major theoretical background on the description and solution to certain types of multi-player games. We will demonstrate that Neumann's and Morgenstern's ideas can be applied to analysing three-person games played in redistribution systems. This will allow us to identify and describe the discrimination equilibrium and consequently the commonly acceptable equilibrium, which are both highly relevant for understanding people's real behaviour, including its ethical aspects. Although superseded in many respects, the work by J. Neumann and O. Morgenstern contains certain important points, which have not been considered enough in later research. Let's now focus on how the aforementioned book analyses the issue of negotiations between three persons. The basic case comes in § 21: *The Simple Majority Game of Three Persons*. The following are the most important passages and paragraphs from which they were taken: *"Each player, by a personal move, chooses the number of one of the two other players. Each one makes his choice uninformed about the choices of the two other players. … If two players have chosen each other's numbers we say that they form a* ***couple****. Clearly there will be precisely one couple, or none at all. If there is precisely one couple, then the two players who belong to it get one-half unit each, while the third (excluded) player correspondingly loses one unit. If there is no couple, then no one gets anything… Since each player makes his personal move in ignorance of those of the others, no collaboration of the players can be established during the course of the play."* [8, pp. 222-223.] As thoroughly described, the game may end up either in two players receiving ½ each and the third player -1, or in each player obtaining 0. This is one of the simplest three-person games, yet it can be extended into a more complex one. In § 21.3., the authors stress that *"the game is wholly symmetric with respect to the three players"* [8, p. 224]. This statement will prove very important. The authors are rather specific in claiming that any potential agreement among the players will always be reached outside the basic game (i.e. it would be an outcome of another game). As a follow-up, the authors take the first step and extend the basic (elementary)

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model of the simple majority game of three persons (§ 22.1.2.): *"…let us now consider a game in which each coalition offers the same total return, but where the rules of the game provide for a different distribution. For the sake of simplicity, let this be the case only in the coalition of players 1 and 2, where player 1, say, is favored by an amount ε… If the couple 1,2 forms, then player 1 gets the amount ½+ε, player 2 gets the amount ½–ε ,and player 3 loses one unit. If any other couple forms (i.e. 1,3 or 2,3) then the two players which belong to it get one-half unit each while the third (excluded) player loses one unit. – What will happen in this game? – …Prima facie it may seem that player 1 has an advantage, since at least in his couple with player 2 he gets more ε than in the original, simple majority game. – However, this advantage is quite illusory. If player 1 would really insist on getting the ε in the couple with player 2, then this would have the following consequence: The couple 1,3 would never form, because the couple 1, 2 is more desirable from 1's point of view; the couple 1, 2 would never form, because the couple 2,3 is more desirable from 2's point of view; but the couple 2,3 is entirely unobstructed, since it can be brought about by a coalition of 2,3 who then need pay no attention to 1 and his special desires. Thus the couple 2,3 and no other will form; and player 1 will not get ½+ε nor even one-half unit, but he will certainly be the excluded player and lose one unit. – So any attempt of player 1 to keep his privileged position in the couple 1,2 is bound to lead to disaster for him. The best he can do is to take steps which make the couple 1,2 just as attractive for 2 as the competing couple 2,3. That is to say, he acts wisely if, in case of the formation of a couple with 2, he returns the extra ε to his partner."* [8, p. 226] This point cannot be overstressed. It explains in depth why the players in the winning coalition have to share their payoff equally. If one of them wanted more, he would find himself outside the coalition and end up losing, rather than profiting (in a zero-sum game). We will further extend the model described in the book and address a case of different amounts that can be gained at the expense of the third player if the two remaining players form a coalition. The problem is described in paragraph 22.2., entitled *Coalition of Different Strength* and can be briefly summarised as follows: Let's assume there are amounts *a, b, c* (*a* = what players 2 and 3 may get from player 1, etc.). If player 1 wanted payoff *x*, then players 2 and 3, after subtracting payoff *x,* must be left with more than or as much as players 2 and 3 would obtain from player 1 if 2 and 3 cooperated, i.e. $(c-x)+(b-x)\geq a$. This means $x\leq(-a+b+c)/2$. Thus player 1 may count upon obtaining the maximum payoff of $\alpha=(-a+b+c)/2$, and likewise players B and C may expect obtaining payoffs $\beta=(a-b+c)/2$ or $\gamma=(a+b-c)/2$. [8, p.228] Let us add the following to this brief summary: In every winning coalition, each player allied with either of the other players gets the same payoff. Or, to put it in the language of economic theory, the opportunity costs of forming any possible coalition are equal, based on the player's potential payoff in another coalition. In the next section we will look at how the initial very simple theoretic model of three-player games and their analysis can be further extended by analysis of more complex games containing other, important and real life elements.

2 Basic terms from the redistribution systems theory

In the next section we will concentrate on analyses of games played in redistribution systems. The main difference is that these games are inconstant-sum games. The redistribution systems theory aims at identifying and describing the general features of group behaviour of people in companies, workplaces, teams, institutions, organisations, etc., i.e. in places where people work together and can share the outcome of their joint performance. In redistribution system games, players need to choose (in one way or another) between their own good (i.e. coalition) and the entire system benefit. Most generally, a redistribution system can be defined as one in which the amounts shared by players depend on how players share it, with this dependence well-known and expressible (for example, by equation). The theory of redistribution systems is addressed in a monograph by P. Budinský, R. Valenčík et al. [3] One of the possible interpretations of what can be defined as a redistribution system is as follows: Assume there are three players, named A, B, C. These players perform differently; their respective performances can be expressed as e_1 (performance of player A), e_2 (player B performance), e_3 (player C performance). If each player is rewarded according to his performance, together they will achieve the highest performance $E = e_1+ e_2 +e_3$. If the reward does not correspond to the performance, the overall the performance of the system will decline; the greater the gap between the performance and the reward, the greater the decline. This can be expressed by term $\eta R(x - e_1;\ y - e_2;\ z - e_3)$, where η is a coefficient; R is the function describing the relevant dependence; x, y, z are the payoffs for individual players. We assume that the function R satisfies the distance axioms and is continuous. Payoffs for three players lie in the redistribution area based on the redistribution equation:

$$x+y+z=E-\eta R\left(x-e_1;y-e_2;z-e_3\right) \tag{1}$$

We will further assume that all points in the redistribution area are Pareto efficient, or in other words, that players took advantage of all possible Pareto improvements. [13] describes an interesting interpretation pertaining to the type of game entitled Tragedy of Commons, which is analogous to the Prisoner's Dilemma but designed for multiple players. Let's take farmers in a drought-troubled country and their limited use of water as an example. The matrix includes one of the farmers on the one side and the others on the other side. If all farmers (both the single one and the others) adhere to their agreement and cooperate (which, in this case, means obey the agreed restrictions on the use of water), both groups would get the highest yields per hectare. Collective and unanimous breaching of the agreement (i.e. failing to cut the use of water) would lead to much lower yields on all parts. If the rules are broken by only a single farmer, his yield would be much higher, leaving virtually no impact on the others. If, by contrast, all farmers minus the one individual farmer breach the agreement, his yield will be even lower than if he joined the rest in violating the agreement. If we consider the other players to be rational and striving to maximise their own benefits, then each of them will view the situation from his own perspective and his dominant strategy will be to violate the restrictions and the final result will be disastrous. Let us note that it is the context of these tasks that provoke general aversion to models which are based on the assumption that players are rational and always aim to maximize their benefit. Consequently, it incites the conviction that the real behaviour of people can only be explained if the models are complemented with an ethical dimension, which is viewed as an exogenous element. E. Ostrom offers a different view of addressing this dilemma [9]. Based on extensive empirical results, she presents a solution in the form of self-governance. Common resources can be managed by the community, without central management. A voluntarily established community can spontaneously create an efficient management of common resources. In other words, a voluntarily established community can protect common ownership, distribute the yields of that ownership among its members, and eliminate unentitled parties. Rather than creating mathematical models, by analysing extensive empirical material from various parts of the world E. Ostrom explores social institutions and applies an evolutionary view. We will demonstrate the existence of two types of equilibriums in redistribution systems, which we disclosed here, pertain to the aforementioned issue. Let us look at a three-person game where the players use precious resource and are allowed to distribute it in a manner consistent with the Tragedy of Commons type of game. We will consider a more complex case, when the individual players (farmers) farm under different conditions. Here each use of additional unit of water results in a different payoff for each of the players. From a microeconomic point of view, the maximum common income can be achieved when the marginal income from the last unit of water to be used by any of the players equals the marginal income from the last unit of water of any other player. Now suppose that players share the water at 6:4:2 ratio. If they diverge from that ratio, their common income will be lower – the more they diverge from that water distribution ratio, the lower their income. We may also accept another assumption, which is intuitively obvious and does not narrow the universality of our problem: if they share the water at 6:4:2, their payoffs (coming from income) will be distributed accordingly, and moreover, the farmers' incomes are proportionate to the allocations of water (this proportion does not necessarily have to be linear). Here the general rule of the relationship between the payoffs and the distribution of water will be as follows: what the farmers can distribute among themselves equals the maximum of what they would be able to distribute (i.e. 12) minus the decline in common income, which results from the fact that the players will not share their income according to optimal proportion based on the equality of marginal income (i.e. the proportion of 6:4:2). This dependence is described by the redistribution equation (1). Function R can be, for example, the generally used Euclidean distance (for the ratio of 6:4:2): R[(x-6); (y-4); (z-2)] = √[(x-6)2+(y-4)2+(z-2)2]; Manhattan distance as the sum of the absolute values of the differences in performance and payoffs of the individual players: R[(x-6); (y-4); (z-2)] = |x-6|+|y-4|+|z-2|; Chebyshev distance, which always selects that difference from among the differences in the performance and payoffs of the individual players that appertains to the player with the greatest divergence: R[(x-6); (y-4); (z-2)] = max[(x-6); (y-4); (z-2)]. [3, pp. 51-73] Further on we will only deal with such areas which only contain points representing Pareto optimal situations (this applies to all of those mentioned above), and will search for solutions on that areas.

3 Discrimination equilibrium and commonly acceptable equilibrium

Let us try to define a very simple game (similar to those we mentioned for zero-sum games from the book by J. Neumann and O. Morgenstern). If two players form a coalition, they can distribute among themselves everything the third player would get. The third player obtains the lowest possible payoff (say 0 in our case; however, there are situations in which the player could receive more and even less than 0). The players who have created a two-person coalition will subsequently distribute payoffs in certain proportion among themselves. The major difference between this case and the one described by J. Neumann and O. Morgenstern is that the amount distributed among the players is not constant. Let us consider lines in which the payoff of one of the players equals 0 to be the so called discrimination lines, i.e. the lines of full discrimination of one player by the other two players. They are formed by points shared by the redistribution area with sides determined by two coordinates of x, y, z (Figure 1). Their course is determined by equations:

$$y + z = E - \eta R(0; y - e_2; z - e_3) \tag{2}$$

$$x + z = E - \eta R(x - e_1; 0; z - e_3) \tag{3}$$

$$x + y = E - \eta R(x - e_1; y - e_2; 0) \tag{4}$$

The question is what payoff to allocate to each of the players if they form a two-person coalition (hereinafter referred to as the ‘coalition’) and if they fully discriminate the third player. Let us recall what happens in a zero-sum game: First, we allocated the value corresponding to the ½:½ distribution. Then we admitted the possibility that one of the players might claim more, and proved that, unless he wished to become discriminated, the player could not claim more. Afterwards, we addressed a situation in which what is distributed is determined by whoever forms the coalition, and laid down a distribution rule: in creating a coalition with a player, every player must claim exactly what he would have claimed if he created coalition with another player. The third requirement can be also met in our more general case game of three persons. Just consider the equations that describe the course of the discrimination lines to be a system of three equations with three unknowns. If the function R satisfies the distance axioms, it is continuous and all points in the redistribution area defined by that function are Pareto-optimal. The result of this (in non-negative values) is three points, to which the following applies: The payoff for player A (i.e. x) would be the same in the coalition with player B or C. The payoff of player B (i.e. y) would be the same in coalition with player A or C. The payoff of player C (i.e. z) would be the same in the coalition with player A or B. Let us call the points located on the discrimination lines and complying with the above system of equations the points of discrimination equilibrium (Figure 1).

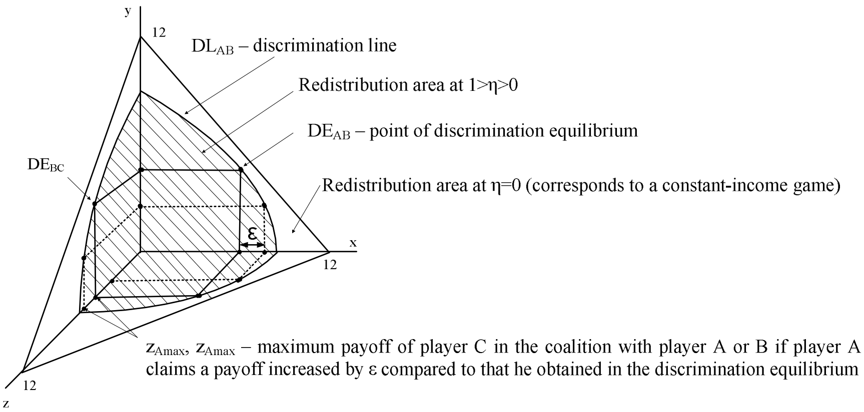

Figure 1 Redistribution area with an indication of discrimination equilibrium

Let the payoff values of players A, B, C, if they are in a coalition that discriminates the third player and if they find themselves in the points of discrimination equilibrium, be called x_d, y_d, z_d and let us suppose that the player outside the coalition will obtain a payoff equal to 0. Let us now introduce the concept of a “better” point in the sense of Neumann-Morgenstern , i.e. point x is better than point y only if the payoff for either of the two players is greater in point x than y. The points of discrimination equilibrium constitute an internally and externally stable set (for a proof, see [3 pp. 49-50]), which complies with [8, § 4.5.3, p. 40, § 32.2., pp. 282-288).]) and can be considered a solution to one type of games played in the set corresponding to the redistribution area. Let us look at another interesting feature of the points of discrimination equilibrium. If player A in a coalition with player B required a payoff by some $\varepsilon > 0$ greater than would be adequate to the discrimination equilibrium, the coalition of players A and B would never be formed – with the reasons being the same as stated in the above-cited passage [8]. Each player may require a payoff for himself only increased by $\varepsilon = 0$. If we considered a more complex game with multiple rounds of negotiations, and wished to express it explicitly (which exceeds the possibilities of our paper), player C would receive more from a coalition with player B than with player A. The reason is that his maximum payoff from a coalition with player A would be z_{Amax} while his possible payoff with player B would be z_{Bmax}, which is necessarily higher because player A claims more than would be adequate to achieve the discrimination equilibrium (Figure 1). Let us reiterate that the positions of all three players in our basic model are fully symmetric. They feel no particular aversion or preference for each other, i.e. the final composition of the coalitions is not influenced from the ‘outside’; it is merely the outcome of what happens in the relevant redistribution system. We ought to make a brief note on what was said above in terms of the practical importance of the models we deal with. If we know what happens in the basic model, which is not subject to any external effects, and compare this to what happens in real-life systems, which we model by means of redistribution systems, we can find various deviations. By analysing these deviations we may subsequently be able to find the external factors (including the hidden ones) influencing the system, which we would not be able to find without the model. This is one of the practical contributions of our activities. However, in this study we explore the basic model which does not reckon with any external effects. What payoff may each player expect? Either the one he gets if he forms a coalition with one of the other two players, or the lowest payoff, which the other players will allocate to him if he is in the discrimination position, i.e. 0. Here it is relevant to state that the player may count upon *the expected average payoff*, which equals $2x_d/3$ for player A, $2y_d/3$ for player B, and $2z_d/3$ for player C (Figure 2). The area marked by the lines of the expected average payoffs has an important feature. Each point inside of it stands for *a Pareto improvement for each player vis-à-vis his average payoff.* This means that instead of an uncertain payoff, though it may be greater than the average expected payoff in two out of three cases but only minimal (equal to 0) in one case, a rationally behaving player is probably going to prefer reaching an agreement on the distribution of payoffs, so that the distribution is consistent with a point inside the considered area. (For the sake of simplicity, we do not consider a decline in the marginal utility of the payoff achieved; if we did, the area of Pareto improvements would be even greater.) Our results are important and may be interpreted in various ways. We can for example say that due to its mathematical foundations reality offers the players better prospects than if they attempt to achieve such results by means of discrimination, or that the mathematical foundations of our reality contain a sort of fairness, our idea of morality, etc. Let us admit, however, that these interpretations use a great deal of fiction. We should also note that the Pareto improvements of the expected average payoffs only exist in inconstant-sum games and games with more than two persons and are not applicable to constant-sum or zero-sum games. However, we can continue to elaborate this and ask the following questions: Which of the points in the area of Pareto improvements is 'the right one', i.e. the one on which the players will agree? Is there any? And if so, can players agree that their payoffs shall be distributed in compliance with it? Issues of this sort are addressed in [1], [2], [5], [6], [7], [10], [11], [12]. One of the possible and convenient answers is Nash's solution to the cooperative game at the set corresponding to the redistribution area and to points defined by ($2x_d/3$; $2y_d/3$; $2z_d/3$) coordinates. In these circumstances Nash's solution brings Pareto improvement which is better than the expected average payoffs. Yet another solution can be taken into consideration, one that we find closer to real life situations. This solution is shown in Figure 2.

Figure 2 will help us answer those questions.

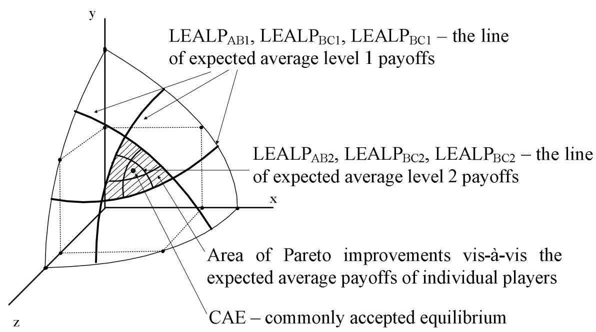

Figure 2 Redistribution area with an indication of the lines of expected average payoffs

Bearing in mind the features/characteristics of the basic game (i.e. a game analogous to the simple majority game) we can imagine that the players will realise that they are better off reaching a mutual agreement. Nevertheless, as concerns the original game, this realisation, as well as the effort for an agreement, if any, is yet another game. Its relationship to the original game is only based on the fact that the exploration of certain features of the basic game has enabled us to unveil the area of Pareto improvements in the expected average payoffs, the possibility of a joint agreement and the issue of defining a game in which the players will be choosing one of the points of those improvements. Each of the players might attempt to form a coalition with any of the other two players in order to achieve the maximum improvement even if all players are already choosing points inside the area of the Pareto improvements of expected average payoffs. The situation would reoccur (compare with [11] for two players), as even the area marked by the expected average payoff lines shares certain features with the redistribution area. It would be possible to define and calculate the points of level 2 discrimination equilibrium, to derive the lines of expected average level 2 payoffs from them, and consequently the area of the level 2 Pareto improvements. Then we could proceed to discrimination equilibriums, the lines of expected average payoffs, the area of level 3 Pareto improvements, etc. The situation resembles a matryoshka doll. Each game which enables definition of an area of Pareto improvements by certain level accommodates another game. These areas gradually become smaller and eventually clearly define one point, *the point of commonly acceptable equilibrium.* Its coordinates correspond to such payoffs the players achieve if they reach a commonly acceptable equilibrium.

4 Issues for further research. Practical application of the research. Conclusion.

It is very probable (and this can be viewed as a hypothesis) that a point of commonly acceptable equilibrium may also be reached by other procedures, e.g. by finding an intersection point of the lines derived from discrimination equilibriums. When searching for the point of commonly acceptable equilibrium, we change parameter d_x, d_y, d_z, (the smallest payoff each player has to obtain) for each player (so far, we have considered that the value of a discriminated player's payoff is 0) from the initial value (i.e. equal to 0) to the maximum possible value (i.e. the absolutely greatest payoff the player can achieve), without changing the values of this parameter for the other two players. Three lines derived from the points of discrimination equilibria will be achieved on the redistribution area. Having established this basis, we can lay down the following two hypotheses: 1. If we change parameter $d_{i,}$ the lines derived from discrimination equilibria intersect in a single point. 2. This point is identical to the point of commonly acceptable equilibrium we have achieved by the above-described negotiations. Further evidence will be required to prove whether this is the same point as the one we found as a result of the 'matryoshka doll' procedure. Several areas are to be addressed by future research, such as defining various types of games played in redistribution system, establishing rules for the players to achieve a commonly acceptable equilibrium through negotiations and thus defining the types of games in which players will fail to achieve a commonly acceptable equilibrium.

For the usual cases of the redistribution area the point of commonly acceptable equilibrium is rather close but not quite identical with the point defined by Nash. No matter how minuscule and insignificant this difference might seem to the players in real-life systems, which share some features with redistribution systems, there are certain existing social objects for which the significance of the above-described seems to be crucial. We called these objects *structures based on mutual covering-up of violation of generally accepted principles*. In the above-mentioned game "Tragedy of Commons" these situations develop when one of the players is caught by the second player when violating their agreement and, as a consequence, is blackmailed. Blackmailing than helps such structures to develop, grow and intertwine with existing social organizations or institutions. Resilience and vitality of these structures can only thrive if the structures manage to develop an inner system with features of a redistribution system in which the players are able to achieve what we called commonly acceptable equilibrium in the three-player game. This is a promising way to detect problems related to corruption and related phenomena.

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