Note on Optimal Paths for Non-Motorized Transport on the Network

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Abstract. The paper presents decision problems concerning the design of routes for non-motorized leisure transport, e.g. cycling or hiking, on the given network. We assume that the network is represented by a non-oriented graph. Three figures are given for each edge of the graph: construction cost, transit time and measure of attractiveness to users. Moreover, the attractiveness measure is defined for each vertex of the graph as well. The problem is to find a path with the duration not exceeding the given limit, maximizing the attractiveness and minimizing the construction cost. The paper describes several alternatives of solution of this two-objective problem.

Keywords: subnetwork, optimal, passenger transport, accessibility, cost.

JEL Classification: C65, R42
AMS Classification: 05C35, 90B06, 90B10

1 Introduction

The main purpose of the paper is to propose mathematical models and methods to support decision making on the choice of economically and 'culturally' optimal route for leisure cycling or hiking (but not both) between two network nodes.

These topics belong to the network economy theory. This scientific field contains two main directions: The first one studies networks of mutually interacting institutions and enterprises, as presented e.g. in the interesting books [5] and [8]. The second direction seeks for economically optimal telecommunication and transport network, see e.g. [14] and [11].

Our problem falls in the family of network reduction problems, namely the ones looking for a subnetwork of the given network. One can mention papers studying similar problems. The paper [6] looks for the maximum planar subgraph of the given graph G or [4] and [9] look for a k-edge connected spanning subgraph which minimizes the edge costs for the given $k> 0$ and similar can be found in [7] and [2] .Minimization of the costs of edges are dealt in [13] or [1] . Other papers seek for maximum cost subgraphs, see. [15] and [1].

As one can see from this survey, no paper from the abovementioned ones deals with non-motorized transport, neither serving for the commuting to work or school, nor for leisure. Therefore, the main contribution of this paper is the formulation of typical problems of leisure cycling and hiking routes design and the presentation of methods solving these problems.

The basic difference between cycle or hiking path for the commuting and the one for leisure ride/walk is that for the trip to work the shortest path is chosen, while for leisure the most attractive route of given yardage (or time) limit is preferred. Note that we use the term 'path' in the sense of 'simple path', which does not pass any intermediate node (=vertex) twice. We use the term 'route' when such multiple passing through vertices or edges is allowed. Even though we shall look for leisure routes, leisure (simple) paths are not a priori excluded.

Assume that there are two terminals of rail or bus transport v_A and v_B. Moreover, suppose that these terminals belong to the vertex set V of the given 'candidate' undirected graph $G = (V, E, a, c, t)$ for either leisure cycling, or hiking, but not both, where:

E is the edge set, each edge $e \in E$ represents a 'candidate' of transformation into the state suitable for leisure hiking or cycling traffic,

a is a non-negative function on the set $E \cup V$, $a(v)$ and $a(e)$ expressing a 'leisure attractivenes' of visiting the vertex $v \in V$ or the edge $e \in E$ respectively, specially $a(v_A) = a(v_B) = 0$,

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c is a positive function on the set E, $c(e)$ expressing the cost of transformation the edge $e \in E$ into the state suitable for leisure hiking or cycling traffic.

t is a positive function on the set E, $c(e)$ expressing the passing time through the edge $e \in E$ for leisure hiking or cycling.

2 Optimization Problems

Problem P1: Let $G = (V, E, a, c, t)$ be the given graph, defined as above. Let $v_A \in V$, $v_B \in V$ be given vertices of G. Let $\tau > 0$ be a time limit for the duration of the trip. The problem is to find a path $p =(v_A = v_0, v_1, ..., v_n = v_B)$ such that

P1.1: $t(p) = t(v_0, v_1) + t(v_0, v_1) + ... + t(v_{n-1}, v_n) \leq \tau$

P1.2: $c(p) = c(v_0, v_1) + c(v_0, v_1) + ... + c(v_{n-1}, v_n) \to \min$

P1.3: $a(p) = a(v_0, v_1) + a(v_1) + a(v_0, v_1) + a(v_2) + ... + a(v_{n-1}) + a(v_{n-1}, v_n) \to \max$

Problem P2: Let $G = (V, E, a, c, t)$ be the given graph, defined as above. Let $A \in V$, $B \in V$ be given vertices of G. Let $\tau > 0$ be a time limit for the duration of the trip and let $\varphi \in (0, 1)$ be an attractiveness reduction factor. The problem is to find a route $r = (v_A = v_0, v_1, ..., v_n = v_B)$ such that $v_{i-1} \neq v_i$ for $i = 1, ..., n$ and

P2.1: $t(r) = t(v_0, v_1) + t(v_0, v_1) + ... + t(v_{n-1}, v_n) \leq \tau$.

P2.2: $c(r) = c(v_0, v_1) + c(v_0, v_1) + ... + c(v_{n-1}, v_n) \to \min$.

P2.3: $a(r) = a(v_0, v_1) + \varphi(v_1)a(v_1) + \varphi(v_1)(v_1, v_2)a(v_0, v_1) + \varphi(v_2)a(v_2) + ... + \varphi(v_{n-1})a(v_{n-1}) + \varphi(v_{n-1}, v_n)a(v_{n-1}, v_n) \to \max$.

There $\alpha(v_i, v_{i+1}) = 1$ if the route r passes through the edge (v_i, v_{i+1}) for the first time and $\alpha(v_i, v_{i+1}) = \alpha$ otherwise, and similarly $\alpha(v_i) = 1$ if the route r passes through the vertex v_i for the first time and $\alpha(v_i) = \alpha$ otherwise.

Although the problem P1 is a particular case of the problem P2, we shall deal with both since they possess different methods of solution.

Note: We use the term 'path' when no vertex is contained more than once in the sequence. If the multiple occurrence of vertices is allowed in the sequence, then we use the term 'route'.

We see that both problems are bicriterial, where the first criterion expresses economic aspects, the second one shows the leisure attractiveness. Theoretically, all known approaches of multicriterial analysis can be applied here but the current Czech situation leads to the applications where limited funding plays the dominant role. Therefore, we suppose a cost limit γ is given and both requirements P1.2 and P2.2 are reformulated to the form:

P1.2'≡ P2.2': $c(r) = c(v_0, v_1) + c(v_0, v_1) + ... + c(v_{n-1}, v_n) \leq \gamma$ ($r = p$ in the case of P1.2')

and then we shall speak about the problems P1' or P2' respectively.

The attractiveness function a is of a combined type, i.e. it is defined as for the edges as for the vertices of the graph. This may cause complication in the solution of the problems. Therefore, we define a new graph with extended vertex and edge sets $G'' = (V'', E'', a, c, t)$, where

$$V'' = (V \cup \{v' : v \in V, a(v) > 0\} \cup \{v'' : v \in V, a(v) > 0\}) - \{v \in V : a(v) > 0\} \tag{1}$$

$$\begin{aligned} E'' = {} & \{(v', v''): v' \in V''\} \cup \{(v, w): v \in V'', w \in V'', (v, w) \in E\} \cup \\ & \cup \{(v, w'): v \in V'', w' \in V'', (v, w) \in E\} \cup \{(v, w''): v \in V'', w'' \in V'', (v, w) \in E\} \cup \\ & \cup \{(v', w'): v' \in V'', w' \in V'', (v, w) \in E\} \cup \{(v', w''): v' \in V'', w'' \in V'', (v, w) \in E\} \cup \\ & \cup \{(v'', w''): v'' \in V'', wC \in V'', (v, w) \in E\} \end{aligned} \tag{2}$$

Moreover, if we define the symbol s as one of the possibilities: single quote (') or double quote (″) or empty space ( ), then we put

$$a(v', v'') = a(v) \text{ for each } v' \in V'' \quad \text{and} \quad a(v^s, w^s) = a(v, w) \text{ for each } (v^s, w^s) \in E'' \tag{3}$$

$$c(v', v'') = 0 \text{ for each } v' \in V'' \text{ and } c(v^s, w^s) = c(v, w) \text{ for each } (v^s, w^s) \in E'', w \neq v \tag{4}$$

$$t(v', v'') = 0 \text{ for each } v' \in V'' \text{ and } t(v^s, w^s) = t(v, w) \text{ for each } (v^s, w^s) \in E'', w \neq v \tag{5}$$

$$a(w) = 0 \text{ for each } w \in V'' \tag{6}$$

Let $r = (v_A = v_0, v_1, ..., v_n = v_B)$ be s route on the graph G, such that $v_{i-1} \neq v_i$ for $i = 1, ..., n$. Then we define the image $r'' = f(r)$ on the graph G'' by replacing of each vertex v_i, having the property $a(v_i) > 0$, by a pair v_i', v_i''.

Lemma 1: f is one-to-one mapping of paths from the graph G onto the ones from G'' preserving values of the functions a, c, and t.

Proof: It is obvious that f is one-to-one mapping. The rest of the proof follows from (3), (4), (5) and (6).

Corollary: Since a path is a particular case of a route, we can solve problems P1' and P2' on the graph G'' instead of G.

Assumption: Throughout the rest of the text, we will assume that the problem P1' is solvable, i e. that there exists a path $p = (v_A = v_0, v_1, ..., v_n = v_B)$ such that $c(p) \leq \gamma$, $t(p) \leq t$.

3 Solution of the Problems P1′ and P2′

In accordance with the Corollary of the Lemma 1 we shall solve the **problem P1′** on the graph G'' where the attractiveness $a > 0$ is assigned only to edges. We shall use an **exact method** of the "Depth-First-Search" type.

The vertex v_A represents the root of the solution tree. All adjacent vertices v to v_A in G'' represent the first level vertices. All adjacent vertices w to all vertices v of the 1st level G'' represent the 2nd level vertices etc.

The search starts in the root v_A with the record $a_{rec} = 0$ and explores each branch as deep as possible. The path p from the root to the current vertex w in the solution tree represents a path in the graph G''. Backtracking is applied when at least one from the following situations occurs:

S1. w is in the path for the second time,

S2. $c(p) > \gamma$,

S3. $t(p) > t$,

S4. $w = v_B$; in that case if $a(p) > a_{rec}$ then the value a_{rec} is increased to $a(p)$.

After having finished the search, the last p is optimal.

Remark: This procedure can serve as a heuristics as well if one stops it before completing the search.

Solution of the **problem P2′** is almost the same as in the case of P1′. Only the first backtracking situation S1 is omitted and $a(r)$ may contain attractiveness of 2nd and further passing through the same edge.

4 Conclusion

In this paper, we have formulated two basic problems concerning the design of leisure routes between two points v_A and v_B for non-motorized tourists, i.e. for hikers or for cyclists (but not for both in the same problem). The first problem concerns simple paths, not passing twice the same point. The second problem deals with routes which are allowed to pass twice or more times through any point.

The path or the route are designed on the given 'candidate' network, where each edge e is assigned three non-negative numbers: $a(e)$ – the attractiveness of passing through e for tourists, $c(e)$ – the construction cost of e and $t(e)$ the passing time through e for a hiker or cyclist. Moreover, the attractiveness $a(v)$ is given for each vertex v. The first problem looks for a (simple) path p connecting the two given vertices v_A and v_B such that the duration $t(p)$ of the passing through the path p does not exceed the given limit t, the total construction cost $c(p)$ of the path p does not exceed the given limit γ and the total attractiveness of p is maximum. The second problem does the same for a general route r which is allowed to pass any vertex or any edge more than once.

A 'Depth-First-Search'-type exact method is presented for both problems and it is noted that the method can be used also as a heuristics if it is stopped before having examined the whole tree of solutions.

The future research can be expected in two main directions. The first one is methodological. As concerns the problem P1, we hope to find a LP formulation of the problem and, moreover, it is likely that there will be found a dual type of heuristics. Maybe, an inspiration can be found in [3], since the problems studied there are similar to P1. Similar expectations can be formulated concerning the problem P2, since it is a bit related to the problem of [10] and [12].

The second direction of the future research will be oriented on the practical experience in applying models and methods described above.

Acknowledgements

The support of the Czech Science Foundation # 402/12/2147 ‘Economically Optimal Processes on Networks’ is gratefully acknowledged.

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