About a modification of $E_r/E_s/1/m$ queueing system subject to breakdowns

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Abstract. The paper deals with modelling of a finite single-server queueing system with a server subject to breakdowns. Customers interarrival times and customers service times follow the Erlang distribution defined by the shape parameter r or s and the scale parameter $r\lambda$ or $s\mu$ respectively. We consider that server failures can occur when the server is either idle or busy (so called operate-independent failures). Further we assume that service of a customer is interrupted by the occurrence of the server failure and the system empties when the server is broken (we call it as the failure-empty discipline). We assume that random variables relevant to server failures and repairs are exponentially distributed. We use two approaches - analytical approach using method of stages and simulation approach using coloured Petri nets. We use both approaches to compare and validation created models. At the end of the paper some reached results are shown.

Keywords: $E_r/E_s/1/m$, queueing, breakdowns, disasters, Petri net.

JEL Classification: C44
AMS Classification: 60K25

1 Introduction

Queuing theory is a tool which enables us to find characteristics of queueing systems. We meet queueing systems for example in informatics, transport and economics. In general, a queueing system represents a system which serves customers coming in the system. For a lot of queuing systems which were solved in the past it is assumed that there are no failures of servers. Such queueing systems are often called reliable queueing systems. The second part of queueing systems are represented by so called unreliable queueing systems or queueing systems subject to server breakdowns.

One of the first queueing systems subject to breakdowns was solved by Avi-Itzhak and Naor [1]. Today we can find a lot of papers solving queuing systems subject to breakdowns. As regards papers written during last 20 years we can mention following papers. Lam et al. [3] modelled a single-server queue with a repairable server under the assumption of the Poisson arrival process and exponentially distributed service times. Tang [7] published the paper devoted to an unreliable single-server queue as well, but with generally distributed service times. Sharma and Sirohi [5] modelled a container unloader as a finite single-server queue with repairable server.

Some queues with several unreliable servers were studied for example by following authors. Martin and Mitrani [4] studied a system with several unreliable servers placed in parallel. Wang and Chang [8] considered a finite multi-server queue with balking, reneging and server breakdowns.

Interesting group of unreliable queueing systems is formed by queues with so-called negative customers or disasters (or catastrophes). Disasters can represent server failures which cause removing either some or all customers finding in the system. We can for example mention the papers written by Boxma et al. [2] or Shin [6].

On the basis of the short review of queueing systems subject to breakdowns we can state that most of the authors studied especially unreliable queueing systems under the assumptions of the exponential or general distribution. Most of the mentioned authors further assumed that a queue of waiting customers has an infinity capacity, if the queue of waiting customers is formed.

In the paper we will pay our attention to a finite single-server queueing system with the server subject to breakdowns, where customers interarrival times and service times will follow the Erlang distribution. Further we will assume that times between failures and times to repair will be exponential random variables. We use the Erlang distribution because it offers us greater variability of usage than the exponential distribution, nevertheless

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mathematical models of queueing systems in which we assume the Erlang distribution are still relatively easily solvable.

The paper is organized as follows. In Section 2 we will make necessary assumptions. In Section 3 we will present the mathematical model, in Section 4 we will present a simulation model of studied queueing system. Section 5 is devoted to the executed numerical experiments and in Section 6 we make some conclusions.

2 General assumptions and notations

Let us study a single server queueing system with a finite capacity equal to *m*, where *m*>1, that means there are in total *m* places for customers in the system – single place in the service and *m*-1 places intended for waiting of customers. Let us consider that customers are served one by one according to the FCFS service discipline.

Let customers interarrival times follow the Erlang distribution with the shape parameter *r* and the scale parameter *rλ*; therefore the mean interarrival time is then equal to $\frac{r}{r\lambda}=\frac{1}{\lambda}$. Costumer service times are an Erlang random variable with the shape parameter *s* and the scale parameter *sμ*; thus the mean service time is equal to $\frac{s}{s\mu}=\frac{1}{\mu}$.

Let us assume that the server is successively failure-free (or available we can say) and under repair. We assume that failures of the server can occur when the server is idle or busy – we say that server failures are operate-independent. Let us assume that times between failures are an exponential random variable with the parameter *η*; the mean time between failures is then equal to the reciprocal value of the parameter *η*. Times to repair are an exponential random variable as well, but with the parameter *ζ*; the mean time to repair is therefore equal to $\frac{1}{\zeta}$. It is clear, that the server steady-state availability *A* (the fraction of time the server is available) is equal to:

$$A=\frac{\zeta}{\eta+\zeta}$$

and the server steady-state unavailability *U* (the fraction of time the server is broken) is:

$$U=1-A=\frac{\eta}{\eta+\zeta}.$$

As regards behaviour of customers at the moment of the failure, we will consider the system empties after every failure of the server; the system is empty when the server is down – failures represent disasters in the system.

3 Mathematical model

To model the studied queueing system we applied method of stages. The method exploits the fact that the Erlang distribution with the shape parameter *r* or *s* and the scale parameter denoted as *rλ* or *sμ* is sum of *s* or *r* independent exponential distribution with the same parameter *sλ* or *rμ*. Therefore the queue can be modelled by Markov chains using.

Let us consider a random variable *K*(*t*) being the number of the customers finding in the system, a random variable *I*(*t*) being the number of terminated phases of customer arrival, a random variable *J*(*t*) being the number of terminated phases of customer service and a random variable *F*(*t*) being the number of broken servers at the time *t*. On the basis of the assumptions established in Section 2 it is clear that {*K*(*t*), *I*(*t*), *J*(*t*), *F*(*t*)} constitutes a multi-dimensional Markov process with the state space

$$\Omega=\{(k,i,j,f),k=0,i=0,...,r-1,j=0,f=0,1\}\cup\{(k,i,j,f),k=1,...,m,i=0,...,r-1,j=0,...,s-1,f=0\}.$$

The system is found in the state (*k*,*i*,*j*,*f*) at the time *t* if *K*(*t*)=*k*, *I*(*t*)=*i*, *J*(*t*)=*j* and *F*(*t*)=*f*, let us denote the corresponding probability $P_{(k,i,j,f)}(t)$.

Let us illustrate the queueing model graphically as a state transition diagram (see figure 1). The vertices represent the particular states of the system and oriented edges indicate the possible transitions with the corresponding rate. Please notice that in figure 1 there are depicted only selected states that are necessary for formation of the equation system.

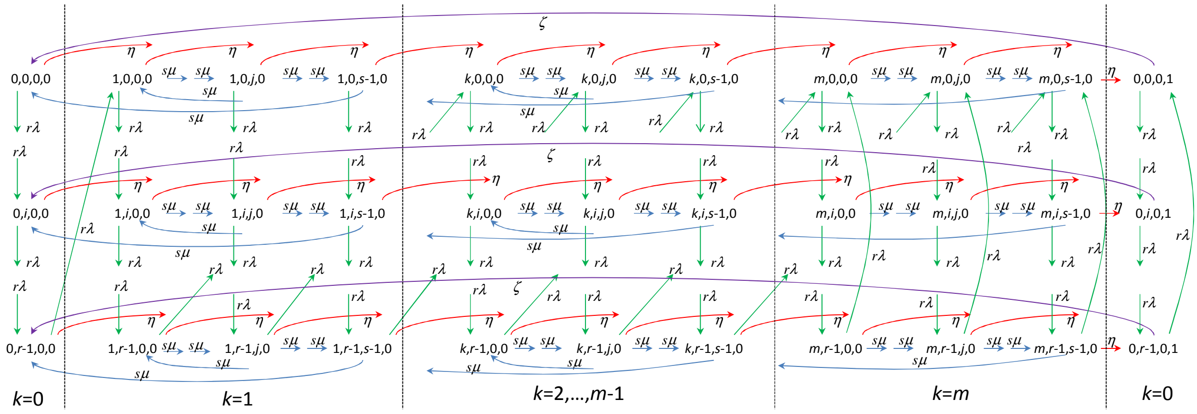

Figure 1 The state transition diagram

On the basis of the state transition diagram we are able to write a finite linear equation system of steady-state balance equations in the form:

$$(r\lambda+\eta)\cdot P_{(0,0,0,0)} = s\mu\cdot P_{(1,0,s-1,0)} + \zeta\cdot P_{(0,0,0,1)}\,, \tag{1}$$

$$(r\lambda+\eta)\cdot P_{(0,i,0,0)} = r\lambda\cdot P_{(0,i-1,0,0)} + s\mu\cdot P_{(1,i,s-1,0)} + \zeta\cdot P_{(0,i,0,1)} \text{ for } i=1,\dots,r-1\,, \tag{2}$$

$$(r\lambda+s\mu+\eta)\cdot P_{(k,0,0,0)} = r\lambda\cdot P_{(k-1,r-1,0,0)} + s\mu\cdot P_{(k+1,0,s-1,0)} \text{ for } k=1,\dots,m-1\,, \tag{3}$$

$$(r\lambda+s\mu+\eta)\cdot P_{(k,i,0,0)} = r\lambda\cdot P_{(k,i-1,0,0)} + s\mu\cdot P_{(k+1,i,s-1,0)} \text{ for } k=1,\dots,m-1, i=1,\dots,r-1\,, \tag{4}$$

$$(r\lambda+s\mu+\eta)\cdot P_{(1,0,j,0)} = s\mu\cdot P_{(1,0,j-1,0)} \text{ for } j=1,\dots,s-1\,, \tag{5}$$

$$(r\lambda+s\mu+\eta)\cdot P_{(k,i,j,0)} = r\lambda\cdot P_{(k,i-1,j,0)} + s\mu\cdot P_{(k,i,j-1,0)} \text{ for } k=1,\dots,m, i=1,\dots,r-1, j=1,\dots,s-1\,, \tag{6}$$

$$(r\lambda+s\mu+\eta)\cdot P_{(k,0,j,0)} = r\lambda\cdot P_{(k-1,r-1,j,0)} + s\mu\cdot P_{(k,0,j-1,0)} \text{ for } k=2,\dots,m-1, j=1,\dots,s-1\,, \tag{7}$$

$$(r\lambda+s\mu+\eta)\cdot P_{(m,0,0,0)} = r\lambda\cdot P_{(m-1,r-1,0,0)} + r\lambda\cdot P_{(m,r-1,0,0)}\,, \tag{8}$$

$$(r\lambda+s\mu+\eta)\cdot P_{(m,0,j,0)} = r\lambda\cdot P_{(m-1,r-1,j,0)} + r\lambda\cdot P_{(m,r-1,j,0)} + s\mu\cdot P_{(m,0,j-1,0)} \text{ for } j=1,\dots,s-1\,, \tag{9}$$

$$(r\lambda+s\mu+\eta)\cdot P_{(m,i,0,0)} = r\lambda\cdot P_{(m,i-1,0,0)} \text{ for } i=1,\dots,r-1\,, \tag{10}$$

$$(r\lambda+\zeta)\cdot P_{(0,0,0,1)} = r\lambda\cdot P_{(0,r-1,0,1)} + \eta\cdot P_{(0,0,0,0)} + \eta\cdot\sum_{k=1}^{m}\sum_{j=0}^{s-1} P_{(k,0,j,0)}\,, \tag{11}$$

$$(r\lambda+\zeta)\cdot P_{(0,i,0,1)} = r\lambda\cdot P_{(0,i-1,0,1)} + \eta\cdot P_{(0,i,0,0)} + \eta\cdot\sum_{k=1}^{m}\sum_{j=0}^{s-1} P_{(k,i,j,0)} \text{ for } i=1,\dots,r-1 \tag{12}$$

including normalization equation in the form:

$$\sum_{i=0}^{r-1}\sum_{f=0}^{1} P_{(0,i,0,f)} + \sum_{k=1}^{m}\sum_{i=0}^{r-1}\sum_{j=0}^{s-1} P_{(k,i,j,0)} = 1\,. \tag{13}$$

We have the equation system of $m\cdot r\cdot s+2r+1$ linear equations formed by equations (1) up to (13) with $m\cdot r\cdot s+2r$ unknown stationary probabilities. To solve it we can omit for example equation (1). Solving of the system we executed using Matlab. Applied state description in the form of (*k*,*i*,*j*,*f*) is four-dimensional and is very good for formation the equation system but is unsuitable for computations in Matlab. Therefore we established an alternative one-dimensional state description in the following form:

- The states (*k*,*i*,*j*,*f*) for *k*=1,...,*m*, *i*=0,...,*r*-1, *j*=0,...,*s*-1 and *f*=0 can be denoted using a single value $(k-1)\cdot r\cdot s+j\cdot r+i+1$,
- The states (*k*,*i*,*j*,*f*) for *k*=0, *i*=0,...,*r*-1, *j*=0 and *f*=0,1 can be denoted using a single value $m\cdot r\cdot s+f\cdot r+i+1$.

Applying the alternative one-dimensional state description we are able to transform the equation system in the form we need for using Matlab (we need a transition matrix). After numerical solving of the equation system rewritten in matrix form we obtain the stationary probabilities we need in order to compute performance measures of the studied system. Let us consider three performance measures – the mean number of the customers in the service *ES*, the mean number of the customers waiting in the queue *EL* and the mean number of the broken servers *EP*.

For the performance measures we can write following formulas:

$$ES = \sum_{k=1}^{m} \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} P_{(k,i,j,0)} ,$$

$$EL = \sum_{k=2}^{m} (k-1) \cdot \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} P_{(k,i,j,0)} ,$$

$$EP = \sum_{i=0}^{r-1} P_{0,i,0,1} = U .$$

4 Simulation model

In order to validate the outcomes, which were reached by solution of the above-mentioned mathematical model, Petri net model of the studied queueing system was created by using CPN Tools – Version 3.0.4. The software CPN Tools is designed for editing, simulating and analyzing coloured Petri nets. The created simulation model in initial marking is shown in figure 2. The model is compound of 11 places and 10 transitions. The Petri net presented in figure 2 models the unreliable $E_r/E_s/1/5$ queueing system fulfilling the conditions mentioned in Section 2. In figure 2 the applied values of individual parameters are r=2, $r\lambda$=18 h^{-1}, s=2, $s\mu$=20 h^{-1}, η=0.1 $h^{-1}$and ζ=0.2 h^{-1}.

The concrete values of the random variables are generated during the simulation through the defined function *fun ET(k, mi) = round(erlang(k, mi/3600.0))*; where *k* is the shape parameter and *mi* is the scale parameter of the Erlang probability distribution expressed in [h^{-1}]. We apply a second as the unit of time.

The created Petri net works with following tokens:
- Timed tokens c represent customers.
- Timed tokens f represent failures of the server.
- Auxiliary tokens p serve for modelling for example free queue places or free servers.

To obtain desired simulation outcomes three monitoring functions were defined:
- The monitoring function named *ES* which is bound with the place *Busy servers* enables estimation of the mean number of the customers in the service.
- The monitoring function named *EL* which is bound with the place *Queue* serves for estimation of the mean number of the waiting customers.
- The monitoring function named *EP* which is bound with the place *Repairing* was create in order to estimate the mean number of the broken servers.

In order to stop the simulation after reaching defined simulation time we created a breakpoint function which stops each simulation run after reaching the simulation time 31 536 000 s; the value corresponds to 365 days. And finally, to execute 30 simulation runs for each experiment we defined an auxiliary text *CPN'Replications.nreplications 30*. Evaluating the text defined number of simulation runs is performed and simulation outcomes gained.

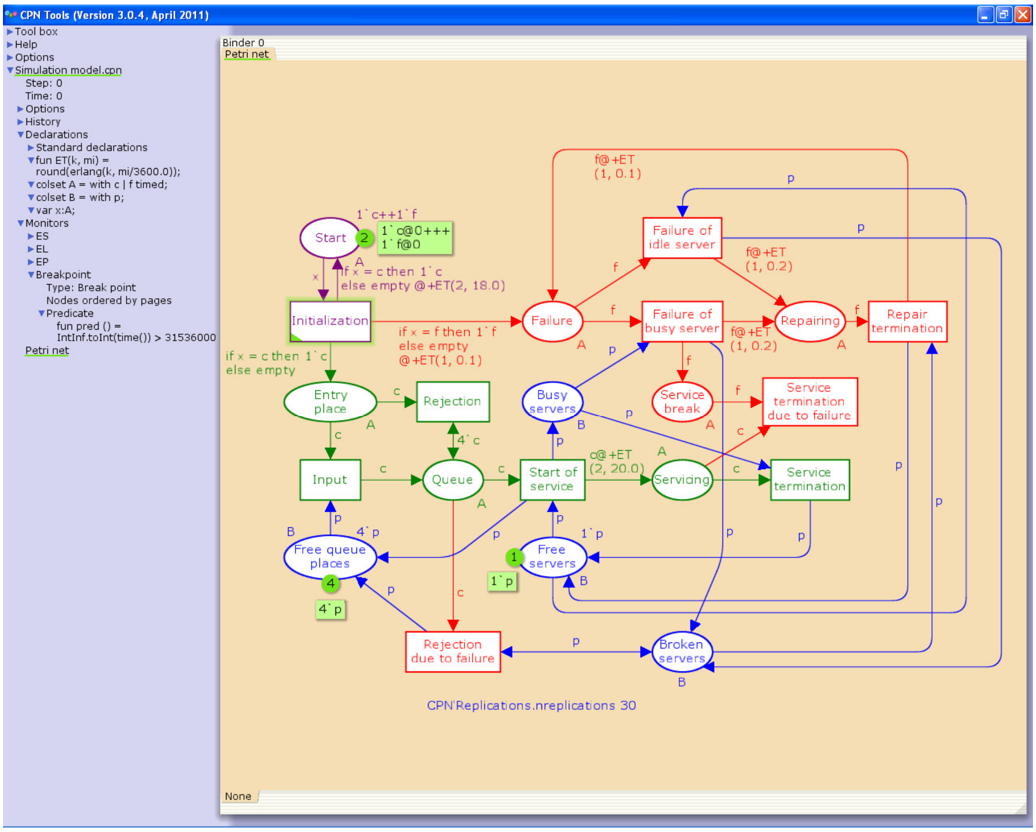

Figure 2 Simulation model created in CPN Tools – Version 3.0.4

5 Outcomes of executed experiments

We executed several experiments with both models to compare analytical and simulation outcomes and to obtain some graphical dependencies. Applied values of model parameters are summarized in table 1.

Parameter	m [-]	r [-]	$r\lambda$ [h^{-1}]	s [-]	$s\mu$ [h^{-1}]	η^{-1} [h]	ζ [h^{-1}]
Applied value	5	2	18	2	20	10 up to 200 with step 10	0.2

Table 1 Applied values of model parameters

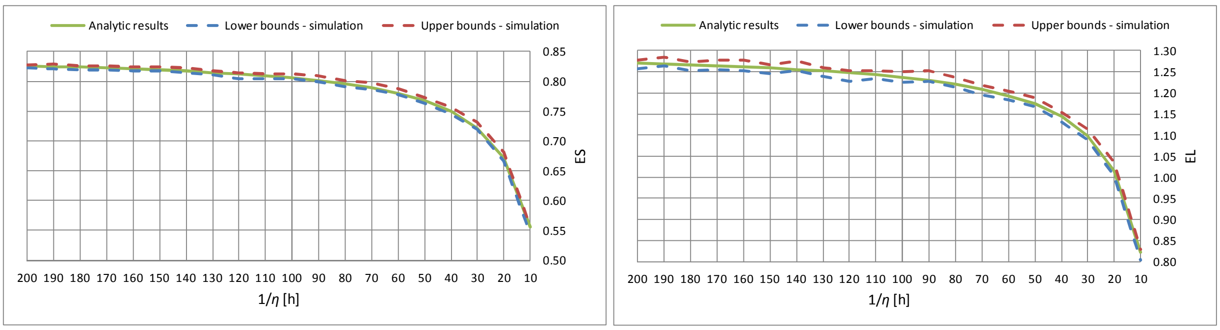

Figure 3 The graphical dependency of *ES* and *EL* on the reciprocal value of η

In table 2 we can see the comparison of analytical and simulation outcomes. On the basis of executed simulations we computed 95% confidence intervals for *ES*, *EL* and *EP* where T_l is the lower bound and T_u is the upper bound of the interval. We can see that for all executed experiments the analytical value of individual performance measures lies in the corresponding confidence interval. We see that both models gives very similar outcomes, we validated the mathematical model.

The graphical dependencies of *ES* and *EL* on η^{-1} are shown in figure 3. As can be seen, both dependencies are decreasing; the fact could be logically expected because increasing η (or decreasing η^{-1}) means more frequent failures that cause lower fraction of time in which the server is able to serve incoming customers and the system is often empty.

η^{-1}	Analytical results			Simulation results					
	ES	*EL*	*EP*	T_l for *ES*	T_u for *ES*	T_l for *EL*	T_u for *EL*	T_l for *EP*	T_u for *EP*
200	0.82593	1.27017	0.02439	0.82170	0.82775	1.25782	1.27871	0.02220	0.02892
190	0.82483	1.26835	0.02564	0.82103	0.82869	1.26434	1.28446	0.02316	0.03079
180	0.82362	1.26633	0.02703	0.81918	0.82614	1.25314	1.27232	0.02375	0.03171
170	0.82226	1.26408	0.02857	0.81879	0.82553	1.25546	1.27814	0.02601	0.03242
160	0.82075	1.26156	0.03030	0.81826	0.82474	1.25185	1.27700	0.02592	0.03208
150	0.81903	1.25872	0.03226	0.81701	0.82351	1.24636	1.26739	0.02692	0.03403
140	0.81708	1.25548	0.03448	0.81457	0.82254	1.25267	1.27641	0.02935	0.03833
130	0.81484	1.25177	0.03704	0.81089	0.81695	1.23865	1.25899	0.03467	0.04087
120	0.81224	1.24746	0.04000	0.80426	0.81486	1.22653	1.25160	0.03764	0.04836
110	0.80920	1.24240	0.04348	0.80450	0.81252	1.23368	1.25310	0.03924	0.04820
100	0.80557	1.23637	0.04762	0.80425	0.81242	1.22531	1.25105	0.03994	0.04770
90	0.80117	1.22908	0.05263	0.79867	0.80864	1.22795	1.25334	0.04702	0.05719
80	0.79575	1.22008	0.05882	0.79148	0.80062	1.21378	1.23613	0.05415	0.06423
70	0.78887	1.20868	0.06667	0.78569	0.79764	1.19535	1.21796	0.05479	0.06839
60	0.77988	1.19377	0.07692	0.77824	0.78744	1.18335	1.20488	0.06809	0.07870
50	0.76762	1.17345	0.09091	0.76243	0.77290	1.16723	1.18747	0.08532	0.09770
40	0.74992	1.14411	0.11111	0.74482	0.75587	1.13000	1.15255	0.10409	0.11654
30	0.72211	1.09804	0.14286	0.71973	0.73204	1.08981	1.11331	0.13153	0.14577
20	0.67207	1.01525	0.20000	0.66656	0.68169	1.00384	1.03585	0.18984	0.20624
10	0.55541	0.82288	0.33333	0.54409	0.55807	0.80376	0.82949	0.33059	0.34610

Table 2 Outcomes of executed experiments

6 Conclusions

In the paper we presented the mathematical model of $E_r/E_s/1/m$ queueing system subject to disasters which causes loss of all customers finding in the system and rejection of all customers incoming to the system while the server is under repair. The mathematical model was solved using Matlab to get the stationary probabilities of system states. The probabilities we need for computing performance measures. We focused on three performance measures *ES*, *EL* and *EP*. The analytic outcomes were validated using simulation; we created a simple simulation model of the system using coloured Petri net. As regard our future research we would like to extend the model; there are another performance measures we are interested in. For example we would like to compute the probability that customer will be rejected upon his arrival or due to the failure of the server.

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