Output analysis and stress testing for mean-variance efficient portfolios

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Abstract. Solutions of portfolio optimization problems are often influenced by model misspecifications or by errors due to approximation, estimation and incomplete information. The obtained results, recommendations for the risk and portfolio manager should be then carefully analyzed. We shall focus on output analysis and stress testing with respect to uncertainty or perturbations of input data for the Markowitz mean-variance model with a general polyhedral convex set of considered portfolios and we shall discuss its robust versions. Possible extensions to general mean-risk efficient portfolios will be delineated.

Keywords: Markowitz model, mean-variance efficient portfolios, stability, stress testing, worst-case analysis.

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AMS classification: 91G10, 90C31

1 The Markowitz model

In conclusions of his famous paper [14] on portfolio selection, Markowitz stated that "what is needed is essentially a 'probabilistic' reformulation of security analysis". He developed a model for portfolio optimization in an uncertain environment under various simplifications. It is a static, single period model which assumes a frictionless market. It applies to small rational investors whose investments cannot influence the market prices and who prefer higher yields to lower ones and smaller risks to larger ones. Let us recall the basic formulation: The composition of portfolio of I assets is given by weights of the considered assets, $x_i, i = 1, \dots, I$, $\sum_i x_i = 1$. The unit investment in the i-th asset provides the random return ρ_i over the considered fixed period. The assumed probability distribution of the vector ρ of returns of all assets is characterized by a known vector of expected returns $E\rho = \mu$ and by a fixed covariance matrix $V = [\text{cov}(\rho_i, \rho_j),\ i, j = 1, \dots, I]$ whose main diagonal consists of variances of individual returns. This allows to quantify the "yield from the investment" as the expectation $\mu(x) = \sum_i x_i\mu_i = \mu^\top x$ of its total return and the "risk of the investment" as the variance of its total return, $\sigma^2(x) = \sum_{i,j} \text{cov}(\rho_i, \rho_j)x_i x_j = x^\top V x$. According to the assumptions, the investors aim at maximal possible yields and, at the same time, at minimal possible risks – hence, a typical decision problem with two criteria, "max" $\{\mu(x), -\sigma^2(x)\}$ or "min" $\{-\mu(x), \sigma^2(x)\}$. The mean-variance efficiency introduced by Markowitz is fully in line with general concepts of multiobjective optimization. Accordingly, mean-variance efficient portfolios can be obtained by solving various optimization problems such as

$$\min_{x\in\mathcal{X}}\{-\lambda\mu^\top x + 1/2x^\top V x\} \tag{1}$$

where the value of parameter $\lambda \geq 0$ reflects investor's risk aversion. Another possibility, favored in the practice, is to minimize the portfolio variance subject to a lower bound for the total expected return, i.e.

$$\min_{x\in\mathcal{X}} x^\top V x \text{ subject to } \mu^\top x \geq k \tag{2}$$

with parameter k, or to maximize the expected return under a constraint on portfolio variance

$$\max_{x\in\mathcal{X}} \mu^\top x \text{ subject to } x^\top V x \leq v. \tag{3}$$

In the classical theory, the set $\mathcal{X} = \{x \in \mathbb{R}^I : \sum_i x_i = 1\}$ without nonnegativity constraints, which means that short sales are permitted. Under this simplification explicit forms of optimal solutions can

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be obtained, e.g. the optimal solution of (1) $x(\mu, V; \lambda)$ is linear in μ. We shall allow for general convex polyhedral sets $\mathcal{X}$. To trace out the mean-variance frontier, one may solve (1), (2) or (3) for many different values of λ, k, v respectively, or to rely on parametric programming techniques, cf. [16]. Notice that the same set of mean-variance solutions is obtained when using $\sqrt{x^\top Vx}$ at the place of $x^\top Vx$.

It was the introduction of risk into the investment decisions which was the exceptional feature of this model and a real breakthrough, and the Markowitz model became a standard tool for portfolio optimization. However, there are many questions to be answered: Modeling the random returns to get their expectations, variances and covariances, the choice of the value of λ, etc. From the point of view of optimization procedures, inclusion of nonnegativity or *linear* regulatory constraints does not cause any serious problems. This however does not apply to minimal transaction unit constraints which introduce 0-1 variables. In the interpretation and application of the results one has to be aware of the model assumptions (not necessarily fulfilled in real-life), namely, that it is a one-period model based on the buy-and-hold strategy applied between the initial investment and the horizon of the problem so that decisions based on its repeated use over more that one period can be far from a good, suboptimal dynamic decision, cf. [3]. See also [17] and references therein for a discussion and multiperiod extensions.

2 Output analysis for the Markowitz mean-variance model

The optimal solution $x(\mu, V\,; \lambda)$ and the optimal value $\varphi(\mu, V\,; \lambda)$ of (1) depend on μ, V and on the chosen value of λ, and at the same time, one can hardly assume full knowledge of these input values. The impact of errors in expected returns, variances and covariances on the optimal return φ of the obtained portfolio was investigated, e.g. in [5]: The program (1) was solved repeatedly with perturbed selected input parameters, ceteris paribus, and the cash equivalent loss was computed for each run. The results of this simulation study indicate that the errors in expected values are more influential than those in the second order moments. Inspired by the cited results we shall deal first with sensitivity analysis of the optimal composition of the portfolio and of the optimal value of (1) on the input values of the expected returns μ of the risky assets and we shall suggest to complement results based on parametric programming by *stochastic sensitivity analysis*.

Assume that the covariance matrix V in (1) is a known *positive definite* matrix, the set $\mathcal{X}$ a nonempty convex polyhedron with nondegenerated vertices, $\lambda > 0$ a chosen parameter value, and that the expected return μ is a parameter of the quadratic program (1). The covariance matrix V and the parameter λ will not be indicated in our denotation of the optimal value and of the optimal solution of (1). Under the above assumptions, for each μ, there is a unique optimal solution $x(\mu)$ of (1) and the optimal value function $\varphi(\mu) := \min_{x\in\mathcal{X}}[-\lambda\mu^\top x + \frac{1}{2}x^\top Vx]$ is a concave function. This follows from a more general statement which is a direct consequence of the inequality valid for minimum of a sum:

Proposition 1. *Assume that the objective function $f : \mathcal{X} \times \mathbb{R}^q \to \mathbb{R}$ in the parametric program*

$$\min_{x\in\mathcal{X}} f(x, p)$$

is linear in the parameter p, the set $\mathcal{X}$ is a non-empty convex set which does not depend on p, and an optimal solution $x(p)$ exists for all p. Then the optimal $\varphi(p)$ is a concave function on $\mathbb{R}^q$.

The set of feasible solutions $\mathcal{X}$ of the quadratic program (1) can be decomposed into finitely many relatively open facets that are identified by indices of active constraints; interior of $\mathcal{X}$ and vertices of $\mathcal{X}$ are special cases of these facets. The parametric space $\mathbb{R}^n$ of vectors $p := \lambda\mu$ can be decomposed into finitely many disjoint stability sets linked with the facets by the requirement that for all p belonging to a stability set, the optimal solutions $x(p)$ of the quadratic program (1) lie in the same facet. It is possible to prove (see [2]) that $x(p)$ is continuous on the whole space $\mathbb{R}^n$, is linear on each stability set and differentiable on its interior.

If, however, p belongs to the boundary of a stability set, $x(p)$ looses the differentiability property and is only directionally differentiable. The optimal value function $\varphi(p)$ is piecewise linear – quadratic differentiable concave function of p. These results explain the observed cases of a relative stability of the optimal value and of an extremal sensitivity of optimal solutions on small changes of the vector μ of expected returns: Whenever the initial value of $p = \lambda\mu$ belongs to the boundary of a stability set, arbitrarily small changes in μ can cause transition to one of the neighboring stability sets. It means not only that some other assets are included into portfolio, but different small changes can cause transition to different stability sets. As a result, the composition of the optimal portfolio is regarded unstable. At the same time, the change of the minimal value of (1) is small for small changes of μ.

Illustrative example. Consider the quadratic program minimize $-p_1x_1 - p_2x_2 + 1/2x_1^2 + x_1x_2 + x_2^2$ on the set $\mathcal{X} = \{x_1 \geq 0, x_2 \geq 0 : x_1 + x_2 \leq 1\}$ which corresponds to the risk-adjusted expected return problem (1) with 3 assets where short sales are not permitted. Decomposition of $\mathcal{X}$ into facets Σ_k, $k = 1, \dots, 7$ is depicted in Figure 1a and the corresponding stability sets $\sigma(\Sigma_k), k = 1, \dots, 7$ are drawn on Figure 1b. Consider point $p_1 = p_2 = 1$ on Figure 1b. For this parameter value, the optimal

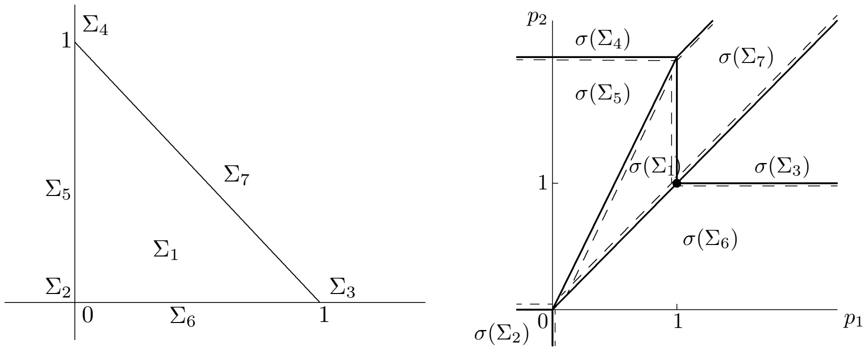

Figure 1: a. Decomposition of the set $\mathcal{X}$ / b. Stability sets

solution is vertex Σ_3, however, a small change of parameter values causes moving the optimal solution into the adjacent facets Σ_6 or Σ_7 or into the interior Σ_1 of $\mathcal{X}$. The corresponding changes of the optimal value and of the first component of the optimal solution are illustrated for fixed $p_2 = 1$ in Figure 2. A similar situation can be observed also in case of changes of the parameter λ (i.e., when tracing the

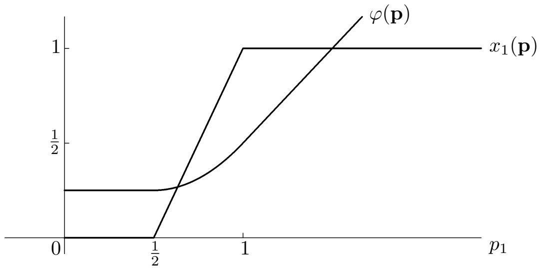

Figure 2: $\varphi(\boldsymbol{p})$ and $x_1(\boldsymbol{p})$ for $p_2 = 1$

mean-variance efficient frontier) but they are more easy to take in as the changes concern only a scalar parameter. Moreover, there is $\lambda_1 > 0$ such that all parameter values $\lambda > \lambda_1$ belong to the same stability set characterized by fixed indices of positive components of efficient portfolio $x(\lambda)$ and that they are linear functions of λ on (λ_1, ∞).

There exist generalizations of the cited results to the case of V positive semidefinite and bounded convex polyhedral set $\mathcal{X}$, hovewer, the fact that from the point of view of quadratic programming there might be multiple optimal solutions indicates clearly the limitations.

Sample based return averages μ_ν are frequently used at the place of the true expectation μ_0. From asymptotic normality of μ_ν, asymptotic normality of the sample optimal value function follows and asymptotic confidence intervals for the optimal value can be constructed.

The optimal, mean-variance efficient solutions $x(\mu)$ are continuous, piece-wise linear, directionally differentiable on certain nonoverlapping convex stability sets in $\mathbb{R}^I$; cf. [2]. Their continuity is sufficient for consistency of the optimal solutions based on a consistent estimate of the true expected return. Except for the simple case of $\mathcal{X} = \{x \in \mathbb{R}^I : \sum_i x_i = 1\}$, $x(\mu)$ are not differentiable. It means that their asymptotic normality holds true only if the true expected return μ_0 lies in the interior of a stability set.

We refer to [8] for details.

3 Stress testing the parameter values

A special type of output analysis appears under the name *stress testing* in the context of quantification of losses or risks which may appear under special, mostly extremal circumstances. Usually, the model is solved or its solution evaluated for an alternative input. We shall indicate now how it is possible to quantify such "stress testing" results.

To stress the parameter values in the Markowitz model we shall apply the contamination technique of robust statistics. In its basic form it requires that the objective function, say $f(x, p)$, is linear with respect to the parameter p and that the set of feasible decisions is fixed, hence, Proposition 1 applies. Changes of parameter p are modeled as $p_t = (1 - t)p + t\hat{p}$ where $\hat{p}$ is a selected parameter perturbation to be tested and $0 \leq t \leq 1$ is a scalar parameter. This approach can be applied to (1) with $\mu_t = (1 - t)\mu + t\hat{\mu}$, $V_t = (1 - t)V + t\hat{V}$ and to (2) or (3) when a *known expected return* μ or a *fixed* covariance matrix V are assumed. To stress separately correlations one can adapt a suggestion of [13]: The covariance matrix can be written as $V = DCD$ with the diagonal matrix D of "volatilities" (standard deviations of the marginal distributions) and the correlation matrix C. Changes in the covariances may be then modeled by "stressing" the correlation matrix C by a positive semidefinite *stress correlation matrix* $\hat{C}$

$$C(\gamma) = (1 - \gamma)C + \gamma\hat{C} \tag{4}$$

with parameter $\gamma \in [0, 1]$. This type of perturbation of the initial quadratic program allows us to apply the related stability results of [2] to the perturbed problem (2)

$$\min_{x\in\mathcal{X}} x^\top DC(\gamma)Dx,\ \gamma \in [0, 1] : \tag{5}$$

where the constraint $\mu^\top x \geq k$ has been incorporaded into the definition of $\mathcal{X}$. Proposition 1 can be specified as

Proposition 2. *Under the above assumptions, the optimal value $\varphi(\gamma)$ of (5) is concave and continuous in $\gamma \in [0, 1]$ and the optimal solution $x(\gamma)$ is a continuous vector in the range of γ where $C(\gamma)$ is positive definite.*

Application of [12], Theorem 17, provides the form of the directional derivative

$$\varphi'(0^+) = x^\top(0)D\hat{C}Dx(0) - \varphi(0).$$

Contamination bounds

$$(1 - \gamma)x^\top(0)DCDx(0) + \gamma x^\top(1)D\hat{C}Dx(1) \leq \min_{x\in\mathcal{X}} x^\top DC(\gamma)Dx$$

$$\leq (1 - \gamma)x^\top(0)DCDx(0) + \gamma x^\top(0)D\hat{C}Dx(0)$$

quantify the effect of the considered change in the input data on the optimal value $\varphi(\gamma)$ of portfolio; cf. [10]. In a similar way, one can quantify the influence of stressing parameters μ, C or μ, V in (1) or parameter μ in (3).

4 Worst-case analysis for the Markowitz mean-variance model

Incomplete knowledge of input data, i.e. of expected returns μ and covariance matrix V may be also approached via the worst-case analysis or robust optimization, cf. [11], [15], [18]. The idea is to hedge against the worst possible input belonging to a prespecified uncertainty or ambiguity set $\mathcal{U}$. We shall denote $\mathcal{M}$, $\mathcal{V}$ considered uncertainty sets for parameters μ and V and will assume that $\mathcal{U} = \mathcal{M} \times \mathcal{V}$. For (1) this means to solve

$$\min_{x\in\mathcal{X}} \max_{(\mu,V)\in\mathcal{U}} \{-\lambda\mu^\top x + 1/2x^\top Vx\}. \tag{6}$$

The worst-case reformulations of (2) and (3) are

$$\min_{x\in\mathcal{X}} \max_{V\in\mathcal{V}} x^\top Vx \text{ subject to } \min_{\mu\in\mathcal{M}} \mu^\top x \geq k, \tag{7}$$

$$\min_{x\in\mathcal{X}}\min_{\mu\in\mathcal{M}} \mu^\top x \text{ subject to } \max_{V\in\mathcal{V}} x^\top V x \le v, \tag{8}$$

respectively. Consider for example $\mathcal{U}$ described by box constraints $0 \le \underline{\mu}_i \le \mu_i \le \overline{\mu}_i$, $i = 1,\dots,I$, $\underline{V} \le V \le \overline{V}$ componentwise and such that V is positive definite. With $\mathcal{X} = \{x \in \mathbb{R}^I : x_i \ge 0\,\forall i, \sum_i x_i = 1\}$ the inner maximum in (6) is attained for $\mu_i = \underline{\mu}_i\,\forall i$ and $V = \overline{V}$. The robust mean-variance portfolio is the optimal solution of

$$\min_{x\in\mathcal{X}}\{-\lambda\underline{\mu}^\top x + 1/2x^\top\overline{V}x\}.$$

We refer to [11] for a survey of various choices of uncertainty sets for the Markowitz model.

5 General mean-risk portfolios

Let us proceed now to static mean-risk models of the Markowitz type with random returns ρ whose probability distribution P does not depend on the selected portfolio composition. (Recall the assumption of a small investor in the Markowitz model.) The yield from the portfolio x is again the expectation $E_P\rho^\top x$, the risk is understood now as a function R which assigns a real number to the uncertain outcome $\rho^\top x$ for the decision x. The value of function R should not depend on the realization of the uncertain return ρ but it depends on the decision and on the probability distribution P; accordingly we shall denote it $R(x, P)$. It should posses some natural properties such as monotonicity, translation equivariance, positive homogeneity and subadditivity for to be called coherent; see [1]. The well-known risk measure Value at Risk (VaR), which is not coherent in general, and the coherent Conditional Value at Risk (CVaR) are special cases of R.

For a known probability distribution P of returns the problems corresponding to (1), (2), (3) are

$$\min_{x\in\mathcal{X}}\{-\lambda E_P\rho^\top x + R(x,P)\}, \tag{9}$$

$$\min_{x\in\mathcal{X}} R(x,P) \text{ subject to } E_P\rho^\top x \ge k, \tag{10}$$

$$\max_{x\in\mathcal{X}} E_P\rho^\top x \text{ subject to } R(x,P) \le v. \tag{11}$$

The form (9) with a probability independent set of feasible decisions is more convenient for applications of quantitative stability analysis techniques, whereas risk management regulations ask frequently for satisfaction of risk constraints with a fixed limit v displayed in (11). Moreover, (11) is favored in practice: solving it for various values of v one obtains directly the corresponding points $[\mu^\top x(v), v]$ on the mean-risk efficient frontier. Numerical tractability of the mean-risk problems depends on the choice of the risk measure and on the assumed probability distribution P. Programs (9)–(11) are convex for convex risk measures $R(\bullet, P)$, such as CVaR; see [6], [10]. As the probability distribution P is fully known only exceptionally, there are two main tractable ways for analysis of the output regarding changes or perturbation of P – quantitative stability analysis with respect to changes of P by stress testing via contamination, see [6], [7], [9], [10], or the worst-case analysis with respect to all probability distributions belonging to an uncertainty set $\mathcal{P}$ which will be briefly discussed below.

The "robust" counterpart of (9) is a straightforward transcription of (6):

$$\min_{x\in\mathcal{X}}\max_{P\in\mathcal{P}}\{-\lambda E_P\rho^\top x + R(x,P)\} \tag{12}$$

whereas for (11) we have

$$\max_{x\in\mathcal{X}}\min_{P\in\mathcal{P}} E_P\rho^\top x \text{ subject to } R(x,P) \le v \text{ for all } P \in \mathcal{P} \text{ or } \max_{P\in\mathcal{P}} R(x,P) \le v. \tag{13}$$

For the Markowitz model, one in fact considers an uncertainty set of probability distributions characterized by fixed expectations and covariance matrices; the Markowitz model does not distinguish among probability distributions belonging to this set. Accordingly, let us specify the class $\mathcal{P}$ as the class of probability distributions identified by fixed moments μ, V known from the Markowitz model. Thanks to the assumed linearity of random returns explicit formulas for the worst-case CVaR and VaR can be derived, cf. [4], and according to Theorem 2.2 of [18] the portfolio composition $x \in \mathcal{X}$ satisfies the worst-case constraint on VaR iff it satisfies the worst-case CVaR constraint.

In general, for convex, compact classes $\mathcal{P}$ defined by moment conditions and for fixed x, the maxima in (12), (13) are attained at extremal points of $\mathcal{P}$. Then under modest assumptions it is possible to pass in (12) and in (13) to discrete distributions $P \in \mathcal{P}$. This convenient property carries over also to $R(x, P)$ that are *convex* in P.

Whereas expected utility functions or CVaR(x, P) are linear in P, various popular risk measures are not even convex in P: the variance is concave in P, the mean absolute deviation is neither convex nor concave in P. This means that extensions of the minimax approach to risk functionals nonlinear in P are carried through only under special circumstances.

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