Microeconomic analysis of cartel equilibrium optimization model

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Abstract. Cartel as a market structure represents a specific form of oligopoly where an agreement is made between legally independent economic subjects in order to restrict the mechanism of economic competition.
Existence of the cartels is in sharp contrast with generally accepted principles and practices of economic competition protection. In developed economies of the EU and the world the governmental institutions are established to control and guarantee the conditions of competition.
In this paper we will present the mathematical formalization of the model of equilibrium price and supply of the cartel agreement participants and point out the social inefficiency of such decision-making scheme. In more detail we will discuss the analysis of the price cartel considering various costs of the producers. We will study the properties of the cartel profit optimization problem considering various marginal costs of homogeneous production of the cartel participants and point out the interpretation possibilities of solving this optimization problem. We will also point out some interesting economically interpretable implications of the Kuhn-Tucker optimality conditions in optimal decision-making of the cartel subjects in the context of its behavior on the market of imperfect competition.ext goes here.

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AMS Classification: 90C30, 90C46

1 Introduction

A cartel is generally perceived as a specific market form of oligopoly where cartel subjects accede to a discreet agreement between formally legally independent economic subjects which together enter into a contract with an aim of reaching a more favorable position on relevant market and thus eliminate the mechanism of competition. Cartels which, based on the agreed market strategies, may follow common price strategy, set their production quotes or divide the market are forbidden in the European Union countries as well as in many countries outside the EU.

A cartel is a specific case of oligopoly with an unspecified number of buyers but only a small number of sellers. The upper limit of the number of the sellers in the market structure for it to be defined as oligopoly is not explicitly defined. Pepall [6] shows, that the key issue is not the number of the sellers but the way they communicate with each other, how they react to their individual intentions and how they jointly address the conditions and attributes of the oligopoly market equilibrium, that is how they resolve the question of market price of their products, how they set the total supply of the sector and how they agree on the individual contribution of the oligopoly subjects to the creation of total oligopoly supply on the relevant market.

It should be noted that each action of a particular firm in an oligopoly affects the behavior of other firms on the market. Price lowering of one firm will likely decrease a market share of other firms on the production of a sector. In other words, responses of the competitors in oligopoly may have a significant effect on a result of managerial decision making on an oligopoly market. It is therefore clear that a decision making about optimal price and supply in an oligopoly is far more complicated than in other market structures.

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In some situations the subjects may simply ignore most of the actions of their competitors, in other cases, however, a price war between the oligopoly subjects may strike as a reaction to seemingly innocent price change. Multiple factors such as maturity of a sector, nature of production and also business methods can determine a way in which firms respond to competitors' behavior.

Methodological problems of formulating oligopoly models rise from the great diversity of ways in which firms can interact and conclude agreements on the distribution, market shares and market prices. Simply said, there is no general model of oligopoly.

In the paper we present the results of microeconomic analysis of a cartel equilibrium model and show how a method of setting an optimal price and supply of a cartel limits the laws of competition. We study the properties of a cartel profit optimization problem considering various marginal costs of homogeneous production of the cartel subjects and point out some interesting, economically interpretable implications of Kuhn-Tucker optimality conditions in a cartel subject's optimal decision making problem in a context of its behavior on the market of imperfect competition.

2 Model of Supply and Price Equilibrium in a Cartel

Let's study a situation in a sector where we assume the individual producers agree on a common course of action while setting a volume of their production, market shares and market price. So the producers will de facto act as a monopoly even though formally they remain independent firms.

If the producers ignored a behavior of the other producers while making their decisions, every firm would independently solved its own profit maximization problem based on classical optimality conditions.

Carlton [1] shows, that if, however, the firms won't proceed independently but agree on common approach to production and sales, by which they violate the conditions of competition, their challenge will then be to find the answers to following questions:

- What will be the equilibrium market state with the cooperating producers in a sector?
- What volume of production will each producer supply?
- What will be a common market price of a product?
- What will be a common profit?
- How will this profit be divided?

So the firms agree on a common course of action. Pepall [6] shows, that the firms will seek such a solution of equilibrium on a market that would maximize their common profit and adjust their individual production strategies to this solution. They will therefore seek optimal volumes of supply of their production based considering common profit function maximization in a form:

$$\pi(Q) = \pi\left(\sum_{i=1}^{n} Q_i\right) = \sum_{i=1}^{n} \pi_i(Q_i) = \sum_{i=1}^{n} TR_i(Q_i) - TC_i(Q_i) \tag{1}$$

where

Q_i – volume of supply of an i th oligopoly producer, $Q_i \in \mathrm{R}$,
$TC_i(Q_i)$ – real function of total costs of an i th oligopoly producer, TC_i: R→R,
$\pi_i(Q_i)$ – real profit function an i th oligopoly producer, $\pi_i(Q)$: R→R,
$TR_i(Q_i)$ – real revenue function an i th oligopoly producer, $TR_i(Q_i)$: R→R,

After expressing a price of cartel production using price-demand function P(Q) in n variables we transform a common profit function maximization problem (1) of cartel subjects to a problem in a form:

$$\pi(Q_1, Q_2, \ldots, Q_n) = \sum_{i=1}^{n} TR_i(Q_i) - TC_i(Q_i) = \sum_{i=1}^{n} \left(P\left(\sum_{i=1}^{n} Q_i\right) Q_i - TC_i(Q_i) \right) \rightarrow \max$$

where

$P(Q)$ – real price-demand function, $P(Q)$: R→R

and after a modification we get:

$$\pi(Q_1, Q_2, \ldots, Q_n) = \sum_{i=1}^{n} \left(P(Q_1, Q_2, \ldots, Q_n) Q_i - TC_i(Q_i)\right) \rightarrow \max \tag{2}$$

Optimization problem (2) represents an unconstrained common profit function maximization problem in n variables and is realized in a stationary point of a concave function of a common cartel profit satisfying necessary optimality conditions in a form:

$$\frac{\partial \pi(Q_1, Q_2, \dots, Q_n)}{\partial Q_i} = \frac{\partial \left(\sum_{i=1}^{n} \left(P(Q_1, Q_2, \dots, Q_n) Q_i - TC_i(Q_i)\right)\right)}{\partial Q_i} = 0 \qquad i = 1, 2, \dots, n \tag{3}$$

A degree of difficulty of a solution of necessary optimality conditions defined by a set of nonlinear equations (3) is, naturally, determined by a degree of difficulty of a price-demand function of a sector and cost functions of particular producers – cartel subjects.

We can show that a market price and a volume of a total supply of a sector set based on the conditions (3) are socially ineffective because they let the firms in a cartel reach higher profits while at the same time have a lower volume of supply and higher market price comparing to other forms of an oligopoly market structure. Regarding this statement let us study cartel profit maximization problem (2) as a mathematical programming problem in a following form:

$$\pi(Q_1, Q_2, \dots, Q_n) = \sum_{i=1}^{n} TR_i(Q_i) - TC_i(Q_i) = \sum_{i=1}^{n} \left(P(Q_1, Q_2, \dots, Q_n) Q_i - TC_i(Q_i)\right) \to \max$$

subject to (4)

$$Q_i \geq 0 \qquad i = 1, \dots, n$$

Let us modify optimization problem (4) to a standard form, i.e. to a problem with an objective function like this:

$$-\pi(Q_1, Q_2, \dots, Q_n) = \sum_{i=1}^{n} \left(-P(Q_1, Q_2, \dots, Q_n) Q_i + TC_i(Q_i)\right) \to \min \tag{5}$$

subject to

$$Q_i \geq 0 \qquad i = 1, \dots, n \tag{6}$$

In [8], we can see that for problems (4), (5) we can formulate generalized Lagrange function in a following form:

$$L(Q_1, Q_2, \dots, Q_n) = \sum_{i=1}^{n} -\left(P(Q_1, Q_2, \dots, Q_n) Q_i + TC_i(Q_i)\right)$$

or

$$L(Q_1, Q_2, \dots, Q_n) = -\sum_{i=1}^{n} P(Q_1, Q_2, \dots, Q_n) Q_i + \sum_{i=1}^{n} TC_i(Q_i) \tag{7}$$

Minoux [5] shows, that Kuhn-Tucker optimality conditions for a Lagrange function (7) of an optimization problem (5), (6) are formulated as follows:

$$\begin{aligned} &\frac{\partial L(Q_1, Q_2, \dots, Q_n)}{\partial Q_i} \geq 0 && i = 1, 2, \dots, n \\ &Q_i \frac{\partial L(Q_1, Q_2, \dots, Q_n)}{\partial Q_i} = 0 && i = 1, 2, \dots, n \\ &Q_i \geq 0 && i = 1, 2, \dots, n \end{aligned} \tag{8}$$

After substitution of a Lagrange function (7) to optimality conditions (8) and after another modification we get the optimality conditions in a form:

$$\begin{aligned} &\frac{-\partial\left(P(Q_i, Q_2, \dots, Q_n)\right)}{\partial Q_i} Q_i - P(Q_i, Q_2, \dots, Q_n) + MC_i(Q_i) \geq 0 && i = 1, 2, \dots, n \\ &Q_i \left(\frac{-\partial\left(P(Q_i, Q_2, \dots, Q_n)\right)}{\partial Q_i} Q_i - P(Q_i, Q_2, \dots, Q_n) + MC_i(Q_i) \right) = 0 && i = 1, 2, \dots, n \\ &Q_i \geq 0 && i = 1, 2, \dots, n \end{aligned} \tag{9}$$

If in the optimality conditions (9) we use own price elasticity of a demand for oligopoly product based on a relation

$$e(Q_i) = \frac{\Delta Q_i}{\Delta P} = \frac{\frac{\partial Q_i}{Q_i}}{\frac{\partial P(Q_i, Q_2, \dots, Q_n)}{P(Q_i, Q_2, \dots, Q_n)}}$$

than in the optimality conditions we can use a following substitution:

$$\begin{aligned} &\frac{-\partial (P(Q_i, Q_2, \dots, Q_n))}{\partial Q_i} Q_i - P(Q_i, Q_2, \dots, Q_n) = \\ &= -P(Q_i, Q_2, \dots, Q_n)\left(\frac{\partial (P(Q_i, Q_2, \dots, Q_n))}{\partial Q_i} \frac{Q_i}{P(Q_i, Q_2, \dots, Q_n)} + 1 \right) = \\ &= -P(Q_i, Q_2, \dots, Q_n)\left(\frac{1}{e(Q_i)} + 1 \right) = -P(Q_i, Q_2, \dots, Q_n) \frac{1 + e(Q_i)}{e(Q_i)} \qquad i = 1,2,\dots,n \end{aligned} \tag{10}$$

and we can finally formulate the optimality conditions in a final form:

$$-P(Q_i, Q_2, \dots, Q_n) \frac{1 + e(Q_i)}{e(Q_i)} + MC_i(Q_i) \geq 0 \qquad i = 1,2,\dots,n \tag{11.1}$$

$$Q_i \left(-P(Q_i, Q_2, \dots, Q_n) \frac{1 + e(Q_i)}{e(Q_i)} + MC_i(Q_i) \right) = 0 \qquad i = 1,2,\dots,n \tag{11.2}$$

$$Q_i \geq 0 \qquad i = 1,2,\dots,n \tag{11.3}$$

Carlton [1] shows, that the scheme would be simpler for a homogenous production and undifferentiated cost functions or the functions of marginal costs of particular producers. Now let's study the properties of cartel profit optimization problem having various marginal costs of cartel participants' homogenous production.

So if the particular producers in a cartel decide on an optimal production Q_i, for i=1, 2, ..., n, meaning that a total cartel supply and a total demand on the market are in balance while maximum of cartel profit function $\pi(Q_1, Q_2, \dots, Q_n)$, then such variables Q_i^* must exist for which the Kuhn-Tucker optimality conditions (24) are satisfied, therefore a vector of variables $(Q_1^*, Q_2^*, \dots, Q_n^*)$ is a solution to a set of equations and inequations (11.1), (11.2), (11.3).

Let us now describe some interesting, economically interpretable consequences of Kuhn-Tucker optimality conditions in a profit maximization problem in the conditions of equilibrium on a cartel market:

1. In the first case let's notice that validity of a condition (11.3) express that in a state of anticipated market equilibrium between aggregated demand and aggregated supply on a cartel market, an optimal value of supply of each cartel producer is either positive or zero. In other words, there may be producers on a market who don't get to supply their production and thus have a zero volume of supply $Q_i^* = 0$. As we show later, this is related with a fact that an equilibrium market price of a cartel covers the marginal costs of a potential cartel participant.
2. Assuming that a cartel participant supplies a positive volume of his production $Q_i^* > 0$, then from a validity of the condition (24.3) results a fact that he must supply such a volume of his production Q_i^* so that a following relation is valid between a cartel market price, his marginal costs and a demand elasticity corresponding to his market supply:

$$P(Q_i, Q_2, \dots, Q_n) = \frac{e(Q_i)}{1 + e(Q_i)} MC_i(Q_i) \tag{12}$$

Let's study an economical interpretation of a relation (12) more closely. We will show that if a company in a cartel supplies a positive volume of production then a market price of a cartel is higher than marginal costs of a firm assuming that a demand for a firm's production is elastic i.e. $e(Q_i^*) < -1$. For an elastic demand for a value of a multiplier $\frac{e(Q_i)+1}{e(Q_i)}$ in (12) stands

$$e(Q_i) < -1 \quad \rightarrow \quad \frac{e(Q_i)+1}{e(Q_i)} > 1 \tag{13}$$

and therefore also

$$P(Q_i, Q_2, \dots, Q_n) > MC_i(Q_i) \tag{14}$$

So we can see that if s firm in a cartel supplies a positive volume of its production, its marginal costs are always lower than an equilibrium market price of a cartel. Fendek [3] shows, that an optimal combination of a price and supply is under stated assumptions for a company in a cartel always more convenient than for a firm in an environment of a perfect competition which compared to (14) must supply such a value of production to equal its marginal costs with a product price.

3. From (12) we can conclude that a bigger share on cartel production is gained by the firms with lower marginal costs because with a cartel common market price they can set bigger volumes of their supply having lower marginal costs. This positive trend of a firm's placement in a cartel is even more significant in a case if higher level of demand elasticity for a firm's production.
4. In a context of above said notes let's discuss economical interpretation of the Kuhn-Tucker optimality condition (11.1). Two situations may occur:
   a. In a case a firm in a point of cartel optimal equilibrium supplies a positive volume of production $Q_i^* > 0$ then from a validity of the condition (11.2) results that a cartel market price is on a level of marginal costs of a firm multiplied by the multiplier from (25) and the optimality condition (24.1) is then realized as an equation and:
   $$-P(Q_i, Q_2, \ldots, Q_n)\frac{1+e(Q_i)}{e(Q_i)} + MC_i(Q_i) = 0$$
   b. In other case, i.e. if the optimality condition (11.1) is realized as a sharp inequality and
   $$-P(Q_i, Q_2, \ldots, Q_n)\frac{1+e(Q_i)}{e(Q_i)} + MC_i(Q_i) > 0$$
   then from a validity of the condition (11.2) formally results that a company in a point of cartel optimal equilibrium is not producing, i.e. volume if production is $Q_i^* = 0$. Vivies [7] shows, that this situation occurs because the firm appears to be technologically ineffective as it has high marginal costs so these after being multiplied by the multiplier from (12) are not covered by a cartel market price
   $$P(Q_i, Q_2, \ldots, Q_n) < \frac{e(Q_i)}{1+e(Q_i)} MC_i(Q_i)$$

Kuhn-Tucker optimality conditions for cartel profit maximization problem confirmed a fact that the firms in a cartel, because of the existing barriers to entry and using their agreement on joint course of action in setting an optimal combination of a price and a volume of product supply on a sector market, set the cartel product price above the level of marginal costs and therefore they reach a superior profit.

As an analysis of Kuhn-Tucker optimality conditions has clearly proven, such situation is a result of a fact that the firms acting on a sector market in a cartel structure formally declare their relations being competitive but in fact they act as a monopoly while setting a combination of a market price and a cartel supply, with only one difference of a profit being then divided by a scheme based on comparison of their marginal costs as it was shown by the description of (11.2) Kuhn-Tucker optimality conditions implications.

3 Conclusion

For current developed economies the different forms of imperfect competition are typical as a prevailing type of market structures. Among these a significant position belongs to an oligopoly which is characteristic for most sectors of national economies of developed countries. In an economic theory, a great attention is paid to the study of theoretical concepts of oligopolies behavior also in a context of applying the principles and rules of economic competition on the markets of those sectors where a strong economic subject exists.

That is because if the objective barriers to entry exist on a relevant market, the existence of economic subjects with a dominant market share is clearly connected with a high risk of a dominant position misuse. Legislation for guarantee of competition conditions and elimination of risks of anti-competition practices is created in the developed economies. In Slovakia these tasks are performed by the Antimonopoly Office of the Slovak Republic.

An oligopoly represents a market structure where a limited number of producers operate on a market of a sector. Firms in an oligopoly must respect an existence of their competitors and seek mutual agreement based on different assumptions and schemes of aligning the interests which will be determined by specific characteristics of a sector. In

this competition scheme an existence of an oligopoly in accordance with the rules of economic competition is socially effective and a presence of the competitive relations on a producers' side is naturally effective as well.

A different situation occurs when the firms in an oligopoly which formally declare their independency and the existence of competitive relations in a sector, close a secret agreement on joint process of fixing a volume of supply and price of a product in a sector, which is clearly against the principles of a competition protection. In this case we speak of a cartel. An aim of this practice is an effort to reach an extra profit for cartel subjects.

If the producers in a sector close the agreements based on cartel principles, they significantly limit a quality of a competition and it is an obligation of a state to regulate or eliminate this condition by legislation. In this paper we dealt with methodological tools of microeconomic analysis for fixing an optimal supply and price of a cartel and showed the ineffectiveness of such market structure.

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