Modeling of competition in revenue management

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Abstract. Revenue management (RM) is the art and science of predicting consumer behavior and optimizing price and product availability to maximize revenue. RM models, despite their success and popularity, still remain somewhat simplistic. The most critical flaw is that current RM models are designed under the assumption that demands are independent random variables. The models are readily extended to competitive settings. The paper is devoted to modeling of competition in revenue management. The revenue management problems under competition can be formulated as games. Results from game theory allow to study the existence and uniqueness of equilibrium policies in revenue management games. An approximation algorithm can be used for solving the revenue management problems under competition.

Keywords: network revenue management, competition, game theory.

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1 Introduction

Revenue management (RM) is the art and science of predicting consumer behavior and optimizing price and product availability to maximize revenue (see [6], [8]). RM models, despite their success and popularity, still remain somewhat simplistic. The most critical flaw is that current RM models are designed under the assumption that demands are independent random variables. Many important components such as pricing, capacity management, overbooking, network revenue management, and choice modeling have been extensively studied, competition have not received enough attention. The models are readily extended to competitive settings.

Network revenue management models attempt to maximize revenue when customers buy bundles of multiple resources. The dependence among the resources in such cases is created by customer demand. This class of problems arises for example in airlines, the problem is managing capacities of a set of connecting flights across a network. The basic model of the network revenue management problem is formulated as a stochastic dynamic programming problem whose exact solution is computationally intractable. There are several approximation methods for the problem. The Deterministic Linear Programming (DLP) method is a popular in practice. The DLP method is based on an assumption that demand is deterministic and static.

Depending on the chosen decision variables, competition in revenue management can be categorized as either price-based or quantity-based. Competition in models can be classified according to some characteristics, as static and dynamic, horizontal and vertical, and others. The revenue management problems under competition can be formulated as games. Results from game theory allow to study the existence and uniqueness of equilibrium policies in revenue management games. A comparison between the centralized system and the decentralized system is a traditional topic for studying game-theoretic models. In the paper [4], authors consider horizontal competition over a single-leg flight and vertical competition over a series of connecting fights, assuming low-fare passengers arrive earlier than high-fare passengers. They compare the centralized and decentralized (competitive) settings. In the paper [3], authors have studied airline capacity allocation under competition. Using the concept of demand overflow, they have proposed game-theoretic models based on well-known approximate models for network capacity allocation in the monopolistic setting. They have investigated both the existence and uniqueness of a Nash equilibrium for different game-theoretic models.

The paper is devoted to modeling of competition in network revenue management. The paper analyzes linear approximations of the joint problem of competition and capacity control in network revenue management. The model combines the DLP model with a game model of competition. The rest of the paper is organized as follows. In section 2, network revenue management models and the DLP method are summarized. In Section 3, the DLP network model is extended to include competition. In Section 4, the competition model is analyzed as a generalized Nash game and a Nash game. In the last section, some concluding remarks to possible extensions and to further research are made.

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2 Network revenue management

The quantity-based revenue management of multiple resources is referred as network revenue management. This class of problems arises for example in airline, hotel, and railway management. In the airline case, the problem is managing capacities of a set of connecting flights across a network, so called a hub-and-spoke network. In the hotel case, the problem is managing room capacity on consecutive days when customers stay multiple nights.

Network revenue management models attempt to maximize some reward function when customers buy bundles of multiple resources. The interdependence of resources, commonly referred to as network effects, creates difficulty in solving the problem. The classical technique of approaching this problem has been to use a deterministic LP solution to derive policies for the network capacity problem. Initial success with this method has triggered considerable research in possible reformulations and extensions, and this method has become widely used in many industrial applications. A significant limitation of the applicability of these classical models is the assumption of independent demand. In response to this, interest has arisen in recent years to incorporate customer choice and competition into these models, further increasing their complexity. This development drives current efforts to design powerful and practical heuristics that still can manage problems of practical scope.

The basic model of the network revenue management problem can be formulated as follows (see [6], [8]): The network has m resources which can be used to provide n products. We define the incidence matrix $\mathbf{A} = [a_{hk}]$, $h = 1, 2, \dots, m$, $k = 1, 2, \dots, p$, where

$a_{hk} = 1$, if resource h is used by product k, and

$a_{hk} = 0$, otherwise.

The k-th column of $\mathbf{A}$, denoted $\mathbf{a}_k$, is the incidence vector for product k. The notation $h \in \mathbf{a}_k$ indicates that resource h is used by product k.

The state of the network is described by a vector $\mathbf{c} = (c_1, c_2, \dots, c_m)$ of resource capacities. If product k is sold, the state of the network changes to $\mathbf{c} - \mathbf{a}_k$.

Time is discrete, there are T periods and the index t represents the current time, $t = 1, 2, \dots, T$. Assuming within each time period t at most one request for a product can arrive.

Demand in time period t is modeled as the realization of a single random vector $\mathbf{r}(t) = (r_1(t), r_2(t), \dots, r_p(t))$. If $r_k(t) = r_k > 0$, this indicates a request for product k occurred and that its associated revenue is r_k. If $r_k(t) = 0$, this indicates no request for product k occurred. A realization $\mathbf{r}(\mathrm{t}) = \mathbf{0}$ (all components equal to zero) indicates that no request from any product occurred at time t. The assumption that at most one arrival occurs in each time period means that at most one component of $\mathbf{r}(t)$ can be positive. The sequence $\mathbf{r}(t)$, $t = 1, 2, \dots, T$, is assumed to be independent with known joint distributions in each time period t. When revenues associated with product k are fixed, we will denote these by r_k and the revenue vector $\mathbf{r} = (r_1, r_2, \dots, r_p)$.

Given the current time t, the current remaining capacity $\mathbf{c}$ and the current request $\mathbf{r}(t)$, the decision is to accept or not to accept the current request. We define the decision vector $\mathbf{u}(\mathrm{t}) = (u_1(t), u_2(t), \dots, u_p(t))$ where

$u_k(t) = 1$, if a request for product k in time period t is accepted, and

$u_k(t) = 0$, otherwise.

The components of the decision vector $\mathbf{u}(\mathrm{t})$ are functions of the remaining capacity components of vector $\mathbf{c}$ and the components of the revenue vector $\mathbf{r}$, $\mathbf{u}(\mathrm{t}) = \mathbf{u}(t, \mathbf{c}, \mathbf{r})$. The decision vector $\mathbf{u}(t)$ is restricted to the set

$$U(\mathbf{c}) = \{\mathbf{u} \in \{0,1\}^{\mathrm{n}},\ \mathbf{A}\mathbf{u} \le \mathbf{c}\ \}.$$

The maximum expected revenue, given remaining capacity $\mathbf{c}$ in time period t, is denoted by $\mathrm{V}_t(\mathbf{c})$. Then $\mathrm{V}_t(\mathbf{c})$ must satisfy the Bellman equation

$$V_t(\mathbf{c}) = E\left[\max_{u \in U(\mathbf{c})} \{\mathbf{r}(t)^T \mathbf{u}(t,\mathbf{c},\mathbf{r}) + V_{t+1}(\mathbf{c} - \mathbf{A}\mathbf{u})\}\right] \tag{1}$$

with the boundary condition

$$V_{T+1}(\mathbf{c}) = 0, \forall \mathbf{c}.$$

A decision $\mathbf{u}^*$ is optimal if and only if it satisfies:

$u_j(t, \mathbf{c}, r_j) = 1$, if $r_j \ge V_{t+1}(\mathbf{c}) - V_{t+1}(\mathbf{c} - \mathbf{a}_j)$, $\mathbf{a}_j \le \mathbf{x}$,

$u_j(t, \mathbf{c}, r_j) = 0$, otherwise.

This reflects the intuitive notion that revenue r_k for product k is accepted only when it exceeds the opportunity cost of the reduction in resource capacities required to satisfy the request. The equation (1) cannot be solved exactly for most networks of realistic size. Solutions are based on approximations of various types. There are two important criteria when judging network approximation methods: accuracy and speed.

The first approach is to use a simplified network model, for example posing the problem as a static mathematical program. We introduced Deterministic Linear Programming (DLP) method (see [8]).

The DLP method uses the approximation

$$\max_{\mathbf{x}} \ \mathbf{r}^T\mathbf{x}$$

subject to (2)

$$\mathbf{Ax} \le \mathbf{c}, \quad \mathbf{x} \le \mathbf{D}, \quad \mathbf{x} \ge \mathbf{0},$$

where $\mathbf{D} = (D_1, D_2, \ldots, D_p)$ is the vector of demand over the periods $t, t+1, \ldots, T$, for product k, $k = 1, 2, \ldots, p$, and $\mathbf{r} = (r_1, r_2, \ldots, r_p)$ is the vector of revenues associated with the p products. The decision vector $\mathbf{x} = (x_1, x_2, \ldots, x_p)$ represent partitioned allocation of capacity for each of the p products. The approximation effectively treats demand as if it were deterministic and equal to its mean $E[D]$.

The optimal dual variables, $\boldsymbol{\pi}^{\mathrm{LP}}$, associated with the constraints $\mathbf{Ax} \le \mathbf{c}$, are used as bid prices. The DLP was among the first models analyzed for network RM. The main advantage of the DLP model is that it is computationally very efficient to solve. Due to its simplicity and speed, it is a popular in practice. The weakness of the DLP approximation is that it considers only the mean demand and ignores all other distributional information. The performance of the DLP method depends on the type of network, the order in which fare products arrive and the frequency of re-optimization.

3 Modeling of competition

Modeling of competition among n firms, indexed $i = 1, 2, \ldots, n$, is based on optimization models of DLP type for individual firms. The next step is searching for equilibrium for competing firms.

We make the following standard assumptions (see [3]):

- The prices of all products are fixed for all firms.
- The demand for one product is independent of that for another product.
- The demand for one product from one firm is correlated to the demand for the same product from other firms.
- Each customer is interested only in one particular product.
- Each customer makes a booking request from his preferred firm and with a certain probability, makes another booking request of the same product from another firm if his first booking request is rejected. If his second booking request is also rejected, then he becomes a lost customer to all firms for this time.

Firms sell p classes of products, indexed $k = 1, 2, \ldots, p$, combined from m resources, indexed $h = 1, 2, \ldots, m$. Let $\mathbf{r}^i$ be unit price vector for firm i. Let $\mathbf{c}^i$ be remaining capacity vector for firm i. Let A^i be the resource-product incidence matrix for firm i. Assume that primary demand for firm i is $\mathbf{D}^i$. A rejected customer from firm i makes another booking request for the same product from other firms. Suppose d^{ji} denotes the overflow rate of particular product from firm j to firm i. That is, if a customer, who prefers firm j, is rejected for a booking request for product by firm j, then he would make a booking request of product from firm i with a probability d^{ji}. The total potential demand for firm i is made up from its own primary demand and the overflow demand from other firms, $\mathbf{D}^i + \sum_{j\ne i} d^{ji}\,[\mathbf{D}^j - \mathbf{x}^j]^+$.

Assume that partitioned booking limits for all other firms other than i are given, firm i aims to determine its optimal partitioned booking limits $\mathbf{x}^i$ by solving the following deterministic linear program (DLP):

$$\max_{\mathbf{x}^i} (\mathbf{r}^i)^T\mathbf{x}^i$$

subject to (3)

$$\mathbf{A}^i\mathbf{x}^i \le \mathbf{c}^i, \quad \mathbf{x}^i \le \mathbf{D}^i + \sum_{j\ne i} d^{ji}\,[\mathbf{D}^j - \mathbf{x}^j]^+, \quad \mathbf{x}^i \ge \mathbf{0}.$$

Each firm satisfies its primary demand and then accepts the overflow demand that cannot be satisfied by rival firms. The objective is that firm i maximizes its total revenue. The first constraints states that the capacity on each resource must not be violated. The second constraint specifies that the allocation to all firms for each product must not exceed the demand for this product. The last constraint shows that the booking limits are nonnegative.

There is possible reformulate the problem (3) into an equivalent nonlinear and non-smooth problem, whose feasible set depends only on the partitioned booking limit $\mathbf{x}^i$ of firm i.

Let $\mathbf{s}^i > \mathbf{r}^i$ be a constant vector for any i.

$$\max_{\mathbf{x}^i} (\mathbf{r}^i)^T \mathbf{x}^i + (\mathbf{s}^i)^T \min\left(0, \mathbf{D}^i + \sum_{j \neq i} d^{ji} \left[\mathbf{D}^j - \mathbf{x}^j\right]^+ - \mathbf{x}^i\right)$$

subject to (4)

$$\mathbf{A}^i \mathbf{x}^i \leq \mathbf{c}^i,$$
$$\mathbf{x}^i \geq \mathbf{0}.$$

The vector $\mathbf{x}^i$ is an optimal solution to the problem (3) if and only if $\mathbf{x}^i$ is an optimal solution to the problem (4). The feasible set of the problem (4) is simpler that of the problem (3).

4 Game models

In this section, we introduce generalized Nash games and generalized Nash equilibrium points (see [5]). The relationship between competition models and game models is shown. Next the existence and uniqueness of generalized Nash equilibrium points are studied. An outline of algorithm for solving competition models is given.

The **generalized Nash game** is a non-cooperative game in which each player's admissible strategy set depends on the other players' strategies. Assume that there are n players and each player i, $i = 1, 2, \ldots, n$, controls variables $\mathbf{x}^i$. In fact $\mathbf{x}^i$ is a strategy of the player i.

Let denote by $\mathbf{x}$ the following vector

$$\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n).$$

And let $N = n_1 + n_2 + \ldots + n_n$. Thus $\mathbf{x} \in \mathrm{R}^N$.

Denote by $\mathbf{x}^{-i}$ the vector formed of all players' decision variables except the one of the player i. So we can write

$$\mathbf{x} = (\mathbf{x}^i, \mathbf{x}^{-i}).$$

The strategy of the player i belongs to a strategy set

$$\mathbf{x}^i \in X^i(\mathbf{x}^{-i})$$

which depends on the decision variables of the other players.

Let $f^i(\mathbf{x}^i, \mathbf{x}^{-i})$ be the payoff function for player i when the joint strategy is $\mathbf{x}$. Aim of the player i, given the strategy $\mathbf{x}_0^{-i}$, is to choose a strategy $\mathbf{x}_0^i$ that solves the following optimization problem

$$\max_{\mathbf{x}^i} f^i(\mathbf{x}^i, \mathbf{x}_0^{-i})$$

subject to (5)

$$\mathbf{x}^i \in X^i(\mathbf{x}^{-i}).$$

For any given strategy vector $\mathbf{x}^{-i}$ of the rival players the solution set of the problem (5) is denoted by $S^i(\mathbf{x}^{-i})$.

Thus a vector $\mathbf{x}_0$ is a **generalized Nash equilibrium** if for any i,

$$\mathbf{x}_0^i \in S^i(\mathbf{x}_0^{-i}).$$

Whenever the strategy set of each player does not depend on the choice of the rival players, that is, for any i,

$$X^i(\mathbf{x}^{-i}) = X^i$$

then the non-cooperative game reduces to find $\mathbf{x}_0 \in \prod_i X^i$ that for any i,

$$f^i(\mathbf{x}_0^i, \mathbf{x}_0^{-i}) = \max_{\mathbf{x}^i} f^i(\mathbf{x}^i, \mathbf{x}_0^{-i})$$

subject to (6)

$$\mathbf{x}_0^i \in X^i,$$

that is a **Nash game**.

If there is no joint constraint in the game, then the generalized Nash game and a generalized Nash equilibrium reduce to a traditional Nash game and a Nash equilibrium respectively. The key difference between generalized Nash games and traditional Nash games is that the strategy space for a player may depend on other players' strategies in the former, but not in the latter, although the payoff functions in both types of games are allowed to be functions of other players' strategies.

The network revenue management DLP problems (3) and (4) are models of competition. The feasible set (strategy set) of the problem (3) involves the strategy variables $\mathbf{x} = (\mathbf{x}^i, \mathbf{x}^{-i})$, while the feasible set (strategy set) of the problem (4) only involves the strategy variables $\mathbf{x}^i$. It is easy to observe that the game based on DLP problem (3) for all firms results in a generalized Nash game, while the game based on DLP problem (4) results in a traditional Nash game with non-smooth and nonlinear payoff functions. This non-smooth property may pose difficulties for proposing computational methods for solving games. DLP problems (3) and (4) are equivalent. Therefore, in the context of game theory, a generalized Nash game is converted into a traditional Nash game.

The existence of a Nash equilibrium (or a generalized Nash equilibrium) for Nash games (or generalized Nash games) is an important topic in game theory. Without equilibrium in a game, players do not know what strategy they should take. The uniqueness of the Nash equilibrium is another important topic in game theory. If there is a unique equilibrium, players can choose their strategies without vagueness. The obvious problem with multiple equilibria is that the players may not know which equilibrium will prevail. The main results for the revenue management games are given in Lemma 1, Theorem 1, and Remark 1.

Lemma 1. *Vector* $\mathbf{x}_0$ *is a generalized Nash equilibrium for the generalized Nash game defined by DLP problem* (3) *if and only if* $\mathbf{x}_0$ *is a Nash equilibrium for the Nash game defined by DLP problem* (4).

Proof. The result follows from equivalence between problems (3) and (4) and the definitions of the generalized Nash equilibrium and Nash equilibrium.

Theorem 1. *There exists a generalized Nash equilibrium for the game based on DLP problem* (3).

Proof. The result follows from Lemma 1 and Theorem 1 of [7], which states that a Nash equilibrium exists for a Nash game if the payoff function for each player is concave with respect to their own strategy and continuous with respect to the strategies of all players and the strategy set for each player is convex and compact.

Remark 1. The uniqueness of a generalized Nash equilibrium for the game based on DLP problem (3) is not guaranteed. There is a unique equilibrium result for traditional Nash game (see [1]). In order to obtain the uniqueness of a Nash equilibrium, the payoff function must be twice continuously differentiable with respect to all strategy variables. Since the payoff functions in Nash game based on DLP problem (4) are non-smooth, the uniqueness of the Nash equilibrium is not guaranteed. Result holds with regard to Lemma 1.

It is well known that traditional Nash games are equivalent to variational problems when the payoff function for each player is continuously differentiable and concave to its own strategies. (see [1]). Generalized Nash games are equivalent to quasi-variational inequality problems. Algorithms for solving quasi-variational problems are not common in the literature. A sequential penalty method for a general quasi-variational inequality problem is proposed in [5]. Another simple approximation algorithm is proposed to use for solving generalized Nash games, where it is not applied a penalty approach but DLP problems are solved in each iteration.

Approximation algorithm

Step 1 (Initialization)
Choose the stopping rule parameter ϵ. Let $s = 1$. Choose a starting point

$$\mathbf{x}(s) = (\mathbf{x}^1(s), \mathbf{x}^2(s), \ldots, \mathbf{x}^n(s)).$$

Step 2 (Searching)
For each player i at iteration s+1, finding $\mathbf{x}^i(s+1)$ by solving an optimization DLP problem (3) assuming

$$\mathbf{x}^{-i}(s+1) = (\mathbf{x}^1(s+1), \ldots, \mathbf{x}^{i-1}(s+1), \mathbf{x}^{i+1}(s), \ldots, \mathbf{x}^n(s))$$

is given.

Step 3 (Stopping rule)
If it holds

$$\|\mathbf{x}(s+1) - \mathbf{x}(s)\| \leq \epsilon,$$

then the algorithm terminates and $\mathbf{x}(s)$ is an approximate generalized Nash equilibrium. Otherwise, set $s := s+1$ and go to Step 2.

5 Conclusions

Revenue management (RM) models, despite their success and popularity, still remain somewhat simplistic. Many important components such as pricing, capacity management, overbooking, network revenue management, and choice modeling have been extensively studied, competition have not received enough attention. The models are readily extended to competitive settings.

In the paper, network revenue management models are completed with game models of competition. The approach seems to be useful and promising for next research. There are some possible extensions of the approach and some areas for further research. The approach can be adapted to other network approximations as well (such as RLP, PNLP, and others) (see [2]). The same formulation can be applied to a variety of network bid-price methods. While it is easy to prove the existence of a Nash or generalized Nash equilibrium, it appears difficult to ensure the uniqueness in general. Some special cases should be analyzed. Convergence of the approximation algorithm for generalized Nash games is a challenge for a future investigation.

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