Optimizing Permutation Methods for the Ordinal Ranking Problem

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Abstract. In many real-world decision-making situations and preferential votings objects or alternatives are ranked by experts from the best to the worst with the goal to achieve a group consensus. This setting is called ordinal (consensus) ranking problem (OCRP). Many methods were proposed for the solution of OCRP with an important class of the so called permutation methods such as the consensus ranking model (CRM) or hybrid distance-based ideal-seeking consensus ranking model (DCM), which seek the consensus in the space of all permutations of the order n. The main limitation of these methods is that the number of permutations of order n grows as n!, making solving this problem virtually impossible for n larger than 10. The aim of the article is to propose a new technique which can reduce the number of permutations the methods‘ need to process, lowering their computational complexity and enabling them to deal with permutations of higher orders. The technique is based on the 'distance theorem' derived in the theoretical part of the paper. This theorem postulates the upper limit on the distance between one of the initial preferences (called median) and the consensus. In the empirical part of the paper, Monte Carlo simulations are carried out to examine the technique's behaviour, applicability and efficiency.

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1 Introduction

In business, politics, entertainment or sport we encounter situations involving preferential votings of objects or alternatives. Often, such preferences have structure of *ordinal ranking,* where objects are assigned their rank order from the 1^{st} to the n^{th} place. This is common especially when a set of experts compile various TOP 10, 25, 100, etc., or when some alternatives in a decision-making situation have to be sorted from the best to the worst. In this setting, finding a group consensus is one of the most important and also difficult tasks, which dates back to the late 18^{th} century and works by Borda [2] and Condorcet [3]. As shown by Arrow in [1] it is impossible to find a unique solution (consensus) to all preference ranking problems while meeting a set of some reasonable criteria (e.g. non-dictatorship).

Problems, which have a solution, can be solved through many different methods proposed over the last centuries (see [2, 3]) as well as recently ([5, 10, 13]). Probably the most familiar methods are Condorcet's simple majority rule and Borda-Kendall's method of marks. Modern methods treat preferences as vectors, permutations or matrices, and use distance functions defined on vector or matrix spaces ([4, 11]), but generally with different results ([6, 13]). Important class of these methods constitute so called *permutation methods* such as the *consensus ranking model* (CRM) of Cook and Kress [5] or the *hybrid distance-based ideal-seeking consensus ranking model* (DCM) by Tavana et al. [13]. These methods seek the consensus in the space of all permutations of order *n*. The consensus is defined as a preference (permutation) with the minimal sum of distances to preferences of all DMs. The limitation of these methods is that the number of permutations of order *n* grows as *n*!, making solving this problem virtually impossible for *n* larger than 10.

The aim of the article is to propose a new technique for permutation methods which can significantly reduce the number of permutations in consideration, thus reducing their computational complexity. The technique is based on the 'distance theorem' derived in the theoretical part of the paper. In the empirical part, Monte Carlo simulations of the technique are carried out to examine its behaviour, applicability and efficiency.

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The article is organized as follows. In Section 2, basic concepts are introduced. In Section 3, the distance theorem is derived and the technique based on this theorem is presented in Section 4. Numerical examples are provided in Section 5 and computer simulations are performed in Section 6. Conclusions close the article.

2 Concepts and notation

Let a set of n alternatives be ranked by a set of k decision makers (DMs). Each DM provides the ranking of all alternatives from the 1^{st} to the n^{th} position. Hence, the number of decision makers is equal to the number of preference rankings, i.e., each DM provides only one ranking. All rankings have the same weight. Rankings of alternatives, such as (A, C, D, B), can be converted into (numeric) permutations by a simple correspondence: $A \equiv 1$, $B \equiv 2$, etc., leading to the permutation (1, 3, 4, 2). In this paper, permutations are denoted $\pi_i, i \in \{1,..,k\}$ and the set of all permutations of order n as S_n.

Because permutations of different DMs might be equal, the set of DMs' permutations is in fact a *multiset* (bag) (A, m) with the *underlying set of elements A* and the *multiplicity function* $m : A \to N$.

Furthermore, we assume that there exist a *metric function d* on the set A. Generally, a metric on a set A is a function $d : A \times A \to \mathbf{R}$, which satisfies the following conditions:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$, for $x, z, y \in A$.

Some appropriate metrics for permutations (derived from metric functions on vector spaces) used in this paper are listed below.

- *Kendall's tau distance* ([7, 8]): the number of transposition of adjacent pairs of digits necessary to turn one permutation into other. Also known as *bubble-sort distance*.
- L_p *metric*: distance of two permutations π_i and π_j is given as:

$$d(\pi_i, \pi_j) = \sqrt[p]{\sum_{k=1}^{n} \left|\pi_{ik} - \pi_{jk}\right|^p} \tag{1}$$

For $p = 1$ we obtain the *Manhattan metric* and for $p = 2$ the *Euclidean metric*.

- As permutations can be converted into permutation matrices, also metric functions on matrix spaces analogous to (1) are available. Such l_1 metric is used in DCM and CRM methods.

Definition 1. Let (A, m) be a multiset of permutations given by all DMs, and d be a metric function. Then:

- *Distance between two permutations* $\pi_i \in A$ *and* $\pi_j \in A$ *is denoted* $d(\pi_i, \pi_j)$.
- *Median is a permutation* $\pi_m \in A$ *with the lowest overall distance* $M = \sum_{i=1}^{k} d(\pi_m, \pi_i)$ *to all permutations from A. Generally, there can be more than one median.*
- *Consensus is a permutation* $\pi_r \in S_n$ *with the lowest overall distance* $\sum_{i=1}^{k} d(\pi_r, \pi_i)$ *to all permutations from A. A consensus can belong to A as well, and there can be more than one consensus too.*

n / k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1										
2	1	1									
3	1	2	2	1							
4	1	3	5	6	5	3	1				
5	1	4	9	15	20	22	20	15	9	4	1
10	1	9	44	155	440	1068	2298	4489	8095	13640	21670
…	…	…	…	…	…	…	…	…	…	…	…

n / k (continued)	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**…**
10	32683	47043	64889	86054	110010	135853	162337	187959	2011089	230131	…
…	…	…	…	…	…	…	…	…	…	…	…

Table 1 Distribution of permutations of order n (vertical axis) with k inversions (horizontal axis) (source: [9]).

As settings with more than one median might cause problems, we also define the *well-defined problems* and *ill-defined problems*:

Definition 2. *Well-defined problem with ordinal preference rankings is a problem where there is precisely one median in a bag A, otherwise it is an ill-defined problem.*

For better understanding of the presented concepts and the *distance theorem* in the next section we provide several simple examples in Section 5. We already know that there are $n!$ permutations of order n, and the maximum distance is $\frac{n(n-1)}{2}$. Another important feature of permutations is their distribution with regard to the number of transpositions of adjacent pairs (inversions) to a given permutation, see Table 1. The distribution is symmetric and also asymptotically normal. Generating functions can be found in [9] and [12].

3 The distance theorem

Intuitively, a consensus seems to be 'close' to the median, as median is a permutation 'in the middle' of DMs' preferences. If this is true, it suffices to search through permutations close to the median (with only a few inversions), while permutations more distant can be safely ruled out. This, in turn, might drastically narrow the space S_n we search through. In this section, we provide a theoretical upper boundary on the distance between a median and a consensus using an arbitrary metric function d.

Theorem 1. Let π_m be a median permutation and let π_r be a consensus permutation, let k be the number of permutations and let $M = \sum_{i=1}^{k} d(\pi_m, \pi_i)$. Then $d(\pi_m, \pi_r) \leq \frac{2M}{k}$.

Proof: According to Definition 1, the median is the permutation which satisfies $M = \sum_{i=1}^{k} d(\pi_m, \pi_i) \leq \sum_{i=1}^{k} d(\pi_j, \pi_i) \ \forall j \in \{1,..,k\}$ and for the consensus $\sum_{i=1}^{k} d(\pi_r, \pi_i) \leq M$ holds. Using the triangle inequality in the definition of a metric we get:

$$d(\pi_m, \pi_r) \leq d(\pi_m, \pi_i) + d(\pi_i, \pi_r) \ \forall i \in \{1,..,k\},$$

and after summation over all i we obtain:

$$k \cdot d(\pi_m, \pi_r) \leq \sum_{i=1}^{k} d(\pi_m, \pi_i) + \sum_{i=1}^{k} d(\pi_i, \pi_r) \leq 2M,$$

after dividing k we finally get to the result: $d(\pi_m, \pi_r) \leq \frac{2M}{k}$. Q.E.D.

Definition 2. *The neighbourhood of the median (N) is a space of* $\pi \in N \subseteq S_n$ *such that* $d(\pi_m, \pi) \leq \frac{2M}{k} = r$, *where r is the radius of N.*

Corollary: $\pi_r \in N$.

The theorem narrows the searched space from S_n to N. When radius r is high, the cardinality of subspace N is not significantly lower than that of S_n. However, when r is low, then the subspace N might be so small that the consensus can be easily found even for a very large n. To illustrate the point, consider a situation where $n = 10$, $k = 6$ and $M = 30$. In permutation methods, such as CRM or DCM, a consensus is searched through S_n with $n! = 3{,}628{,}800$ permutations. However, according to the distance theorem, it suffices to search only through permutations closer than or equal to 10 inversions to the median, which is (see Table 1) only about 52,000 cases. Thus, there is a reduction of more than 3,5 million permutations (the factor of 70)!

4 Reduction technique

To allow one to solve problems involving permutations of higher orders using the permutation methods, we propose a technique for reducing the amount of permutations which the methods need to search in order to lower their overall computational complexity. The reduction technique exploits the distance theorem presented in the previous section to narrow the searched space from S_n to N. It alters the computation performed by a permutation method so that it proceeds in the following steps:

1. Preference rankings of n alternatives by k DMs are turned into permutations.
2. Median is found as the permutation(s) with the lowest overall distance to all the preference rankings.
3. If there is more than one median, the problem is ill-defined and the computation stops.
4. The radius r is computed to determine the space (N) which needs to be searched.
5. The efficiency of the reduction is tested and the computation stops if the subspace N is still too large to find the consensus in a reasonable time.
6. A consensus is found in the neighbourhood (N) of median using an arbitrary permutation method.

The metric used in step 2 is arbitrary. The step 3 excludes ill-defined problems explicitly, but cases with two or more medians can be, in general, handled too – by proceeding into step 6 for each median separately. Note that performing the reduction is quite cheap as its computation complexity is n^2k^2, which imposes a negligible overhead compared to the $n!$ iterations a permutation method normally needs to perform, so even if the reduction is not significant, there is no reason not to try it.

5 Numerical examples

In this Section, aforementioned concepts are explained on simple illustrative examples.

Example 1. (trivial) Let all permutations be equal. Then we have: $\pi_i = \pi_m \ \forall i$, $M = 0$, so $d(\pi_m, \pi_r) \le \frac{2M}{k} = 0$ and $\pi_r = \pi_m$ as expected.

Example 2. Suppose that three DMs rank five alternatives A, B, C, D and E as follows: $DM_1 = \pi_1$ = (A, C, E, B, D), $DM_2 = \pi_2$ = (C, A, B, E, D) and $DM_3 = \pi_3$ = (A, B, C, D, E). We compute the distances among all permutations using the Kendall's tau distance: $d(\pi_1, \pi_2) = 2$, $d(\pi_1, \pi_3) = 3$ and $d(\pi_2, \pi_3) = 2$. Median is π_2 (the problem is well-defined), $M = 4$, $r = \frac{2M}{k} = \frac{8}{3}$, so we look for the consensus up to distance 2 from the median (the distance is an integer). This means only 14 permutations (see Table 1) out of 120 permutations have to be tested. The consensus is (A, C, B, E, D).

Example 3. Now consider different situation: $DM_1 = DM_2$ = (A, C, E, B, D) and $DM_3 = DM_4$ = (A, B, C, D, E). Clearly, we have a situation with two opposing groups, where consensus cannot be achieved. It is easy to verify that the median (as well as consensus) is each of the four permutations, hence the problem is ill-defined.

Example 4. Assume that there are the following preference rankings of five objects: (A, B, C, D, E), (B, A, C, D, E), (B, C, A, D, E), (B, C, D, A, E) and (B, C, D, E, A). All permutations changes by one transposition of A. The median is the 'middle' permutation (B, C, A, D, E), $M = 6$, $r = 12/5$, and $\pi_r = \pi_m$.

6 Computer simulations

To test the behaviour, applicability and efficiency of the reduction technique, extensive Monte Carlo simulations were performed for different values of n (number of alternatives) and k (number of DMs) with more than 100,000 randomly generated problems. The analysis was performed by the *OCRP Solver*, a decision support tool available from: *http://www.fit.vutbr.cz/~ifiedor/ocrpsolver/*. The consensus was searched by 3 different methods: DCM, CRM (both briefly described in Section 1) and MED, because the consensus found by different methods might differ in general. The MED method is a simple method which finds a consensus as the permutation(s) with the lowest distance to all other permutations given by DMs. The reason to use it is that it searches the consensus in a similar way as the median is searched, so it might be interesting to find out if the consensus found by this method will be close to the median. The results of simulations are shown in Tables 2-5.

Number of alternatives (n)	4	5	6	7	8	9	10	12	14	16	18	20
Well-defined problems (expressed in percent)	0.46	27.34	64.84	82.11	87.29	89.9	91.87	94.51	95.58	96.62	97.46	97.96

Table 2 A distribution of well-defined problems in OCRP for different n (100 DMs, 10 000 problems).

First, we tested how many problems are well-defined for different n and the results are shown in Table 2. As can be seen, for a larger number of alternatives the number of well-defined problems is growing (presumably to 100%). We also examined whether one median implies a unique consensus, but the answer was negative.

Next, we focused on analysing the distance between a median and a consensus for different values of n. The results are presented in Table 3. From these results, some observations can be made:

- The average radius $r = \frac{2M}{k}$ was getting larger with growing n (and the space S_n which need to be searched).
- The subspace N of S_n was not much smaller than S_n on average (see the $avg\left[|N|/|S_n|\right]$ values expressing the ratio between the reduced space N and the original space S_n in percent). However, the average (empirical) distance between the median and the furthest consensus (the solution might contain more than one consensus if they are all equally good) was much smaller than r (see the $avg\left[d\left(\pi_r,\pi_m\right)/r\right]$ values expressing the ratio between the actual distance $d\left(\pi_m,\pi_r\right)$ and the theoretical maximal distance r, as r differs for each problem), which means that in order to find the consensus a significantly smaller subspace of S_n was enough to search through. For example, for n = 8 and the CRM method, the average distance found between median and the furthest consensus was just $0.2 \cdot avg\left[r\right] = 4.458$. It means that, in average, it would be sufficient to process only the permutations with the distance from median up to 4 and there are only 285 such permutations from all 40,320 permutations. That is a difference in more than two orders of magnitude.

Number of alternatives (n)	$avg\left[r\right]$	$avg\left[\lvert N\rvert/\lvert S_n\rvert\right]$	$avg\left[d\left(\pi_r,\pi_m\right)/r\right]$					
			π_r nearest from π_m			π_r furthest from π_m		
			CRM	DCM	MED	CRM	DCM	MED
4	4.5362	83.33%	0.044894	0.000986	0.160571	0.103846	0.08047	0.29142
5	7.7154	88.33%	0.066591	0.023921	0.272881	0.117663	0.215662	0.357354
6	11.7322	93.19%	0.100863	0.068657	0.317233	0.128777	0.256401	0.389836
7	16.5156	96.55%	0.134482	0.105069	0.339702	0.160821	0.279646	0.402664
8	22.2904	98.44%	0.178254	0.141813	0.370139	0.202144	0.304321	0.431101

Table 3 Average distance between a median and a consensus for different n (10 DMs, 1000 problems).

Number of alternatives (n)	30% of $avg\left[r\right]$	$avg\left[\lvert N_H\rvert/\lvert S_n\rvert\right]$	Number of solutions containing at least one π_r in N_H		
			CRM	DCM	MED
4	1.36086	16.67%	93.85%	100.0%	77.95%
5	2.31462	11.67%	92.49%	99.34%	60.26%
6	3.51966	6.8%	90.92%	97.38%	47.08%
7	4.95468	3.2%	86.60%	96.78%	39.01%
8	6.68712	3.0%	80.58%	93.61%	30.04%

Table 4 The amount of solutions with at least one consensus in space N_H defined by 30% of r for different n.

Of course, we never know if we have found the consensus until we have processed all the permutations in N, so we cannot search just the permutations up to the average distance. But we can use the reduction technique as a heuristic and try to search only a small part of N in order to find the closest permutation to the preferences given by DMs in a reasonable time, even if we do not know if the permutation is the actual consensus or not. It is still better to find a suboptimal solution to the problem than not find any solution at all. Table 4 shows the probability that we will find at least one consensus (out of possibly many equally good) if we search only a subspace of N up to the 30 % of the average r. As can be seen, the reduction is significantly better in this case as we for, e.g., n = 8 search only 3 % of S_n (compared to 98.44 % before), and we still find a consensus in more that 80 % cases.

Number of DMs (k)	Entropy	$avg\left[r\right]$	$avg\left[d\left(\pi_r,\pi_m\right)/r\right]$					
			π_r nearest from π_m			π_r furthest from π_m		
			CRM	DCM	MED	CRM	DCM	MED
10	0.816470	16.51560	0.134482	0.105069	0.339702	0.160821	0.279646	0.402664
5	0.640592	14.60120	0.124172	0.163831	0.319748	0.164463	0.241548	0.740827
3	0.465760	12.18071	0.082859	0.145048	0.285900	0.116300	0.215913	0.966611

Table 5 Average distance between a median and a consensus for different entropy of DMs' preferences (n = 7).

One problem with searching only a small part of N is that with higher n, the average radius r and the distance between the median and consensus is getting higher, decreasing the probability that we will find the consensus in this small part of N. As the problems randomly generated by the Monte Carlo algorithm consist usually from rankings uniformly distributed among the whole space S_n, we are, in fact, dealing with worst case scenarios as real problems normally contain rankings relatively close to each other and not distributed among the whole space S_n. To find out how the average radius r and the distance between the median and consensus will be like when we are more close to the real problems, we examined the relationship between the entropy of DMs‘ preferences and the distance between median and consensus. In [6] the entropy of decision makers’ preferences was introduced as a measure of concordance among DMs. The lower entropy means larger agreement of DMs‘ preferences and vice versa. Our simulations showed that when entropy was lower, the radius r was smaller, and also the empirically found distance between median and consensus was lower on average for DCM and CRM, see Table 5. Based on these results, we believe that for real problems the reduction will be much more effective. Moreover, also the number of well-defined problems was getting larger with the decreasing entropy.

To summarize the results, it was learned that the actual distance between a median and consensus was much smaller than the average radius r (from nearly 0 % to 37 % of r for selected k and n) derived from the distance theorem, which means that the consensus often lies in a close neighbourhood of the median and the number of permutations processed can be reduced significantly. Based on these empirical results, the proposed reduction technique can also be considered an efficient heuristic, especially for problems with similar preferences of DMs (with lower entropy), where the radius r is small and the number of permutations, which must be processed, is low. At last, it should be noted that the technique was tested on randomly generated problems, but real problems are far from being random, so the true test of the technique is possible only by practice.

7 Conclusions

The aim of the article was to introduce and test a new optimization technique for the permutation methods for the solution of the ordinal consensus ranking problem. The technique lowers the computational complexity of these methods, which is NP hard due to $n!$ permutations, by reducing the number of permutations they must process in order to find a consensus. It is based on the distance theorem which postulates the maximal theoretical distance of the consensus from one of the initial preferences (called median). Monte Carlo simulations showed that while the maximal theoretical distance was often too large, the empirically found distance between the median and the consensus was much smaller on average. This allows to use this technique also as an efficient heuristic searching only a small part of the reduced space with a high probability to find a consensus, or permutation close to it, in a reasonable time. Moreover, if the decision makers’ preferences are similar, the maximal theoretical distance and the empirically found distance are getting lower, making the reduction more significant and beneficial.

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