Matrix period in max-drast fuzzy algebra

Martin Gavalec[1], Zuzana Němcová[2]

Abstract. Periods of matrix power sequences in max-drast fuzzy algebra and methods of their computation are considered. Matrix power sequences occur in the theory of complex fuzzy systems with transition matrix in max-t algebra, where t is a given triangular fuzzy norm. Interpretation of a complex system in max-drast algebra reflects the situation when extreme demands are put on the reliability of the system. For triangular norm $t = \min$, the matrix periods have been already described in literature. In this paper, a polynomial algorithm for computing the matrix period in max-drast algebra is presented, and the relation between matrix periods in max-min and max-drast cases is described.

Keywords: drastic triangular norm, max-drast algebra, matrix powers, fuzzy systems.
AMS classification: 08A72, 90B35, 90C47

1 Introduction

Transition matrices with fuzzy values are applied in fuzzy approach to discrete time systems. Various triangular fuzzy norms t are used, in dependence on the character of the system. Development of the system in time is then described by power sequence of the transition matrix, computed in max-t fuzzy algebra. Steady states of the system correspond to periodic behavior of the power sequence.

Max-t fuzzy algebra uses instead of conventional operations for multiplication of vectors and matrices the operations of max and one of the triangular norms - min, drast, prod or Lukasziewicz. Algebras using operations maximum and minimum, possibly in combination with addition or multiplication, were studied by many authors, see e.g. [2], [3], [4], [5], [6], [8], [9], [10]. For triangular norm $t = \min$, the matrix periods are described in [7].

Matrix power sequences and their periodicity in max-drast algebra are studied in this paper and a polynomial method for computation of the matrix period is described. New elements do not appear in the matrices during the computation of matrix powers by operation max and drast. Therefore, some repetition in the matrix power sequence always occurs, which inevitably leads to periodic behavior.

The drastic triangular norm is the basic example of a non-divisible t-norm on any partially ordered set, see [1]. Interpretation of a complex discrete time system in max-drast algebra reflects the situation when extreme demands are put on the reliability of the system. The matrix powers in max-drast algebra behave differently than those in max-min algebra, however the properties of matrix periods in both cases show some similarity described in the paper.

2 Basic notions

In the paper we work with the max-drast fuzzy algebra $(\mathcal{I}, \oplus, \otimes_d)$, where $\mathcal{I}$ is the unit interval $\langle 0, 1\rangle$, $\oplus$ is the maximum operation on $\mathcal{I}$ and $\otimes_d$ is the binary drast operation (drastic triangular norm) on $\mathcal{I}$ defined as follows

$$\mathrm{drast}(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1 \\ 0 & \text{if } \max(x, y) < 1 \end{cases}$$

[1]University of Hradec Králové/Faculty of Informatics and Management, Rokitanského 62, Hradec Králové 3, 500 03, e-mail: martin.gavalec@uhk.cz
[2]University of Hradec Králové/Faculty of Informatics and Management, Rokitanského 62, Hradec Králové 3, 500 03, e-mail: zuzana.nemcova@uhk.cz

Let n be a fixed natural number, we shall use the notation $N = \{1, 2, \ldots, n\}$. Let $A \in \mathcal{I}(n, n)$ be a fixed matrix, the entries of A are denoted as $a_{ij} \in \mathcal{I}$, $i, j \in N$.

If A is interpreted as the reliability matrix of transitions in a discrete time fuzzy system, then the drast norm in the computation of the matrix powers reflects extreme demands on reliability in the system. This follows from the computation of the powers of A. More precisely, zero element $a_{ij} = 0$ of the matrix A means that the reliability of the transition from state i to state j in the system is zero. If $a_{ij} = 1$, then the transition between states i and j is reliable. On the other hand, if value of a_{ij} lies between zero and one, i.e. $a_{ij} = u$, $u \in (0, 1)$, then some uncertainty in the transition expressed by the value u must be considered. The smaller value, the higher degree of uncertainty is expressed.

The idea of reliability of edges (one-step transitions) can be extended to the reliability of paths in the characteristic weighted digraph $G(A)$ of matrix A (several-step transitions). The nodes of $G(A)$ correspond to states of the system and are denoted by numbers in N, while an edge from node i to node j has the weight $w(i, j) = a_{ij}$, for all $i, j \in N$. Getting from state i to state j reliably means that all necessary steps are reliable (weighted by 1). If there is one uncertain step (having positive weight smaller than 1) in the sequence of steps, then the sequence is considered as uncertain. Extreme demands on the reliability of the system imply that any sequence containing two or more uncertain steps is considered as inadmissible.

Formally, a path is called *strong* if all edges of the path are reliable (weighted by 1). The path is called *weak* if and only if exactly one edge of the path is weighted by $u \in (0, 1)$ and the weight of the remaining edges is 1. Finally, the path is called *inadmissible* if more than one edge in the path is weighted by value less than 1, in other words, there is more than one uncertainty on the path from node i to j, or if all edges in the path have zero weight. The set of all strong (weak) paths is denoted by SR (WR). The notation $SR(i, j, r)$ ($WR(i, j, r)$) stands for the set of all paths in SR (WR) from i to j of length r.

3 Matrix powers in max-drast algebra

Definition 1. Let $i, j \in N$, let $p = (i_0, i_1, \ldots, i_r)$, with $i_0 = i, i_r = j$ be a path in the $G(A)$. The weight of path p in max-drast algebra is set to the drastic fuzzy product

$$w_d(p) := \bigotimes_d \{a_{i_{s-1}i_s};\ s = 1, 2, \ldots r\} \ . \tag{1}$$

Let us denote the entries of the r-th power, A^r, of matrix A as a^r_{ij} . The interpretation of this value is given in the following proposition.

Proposition 1. *Let $A \in \mathcal{I}(n, n)$, $i, j \in N$, let r be a natural number. Then the entries of A^r have values given by the formula*

$$a^r_{ij} = \bigoplus \{w_d(p);\ p \text{ is a path in } G(A) \text{ from node } i \text{ to node } j \text{ of length } r\} \ . \tag{2}$$

Proof. The assertion is proved easily by recursion on r, using Definition 1 and the matrix multiplication in max-drast algebra. □

More specific formula for the values of elements in r-th matrix power in max-drast algebra is presented in the next proposition.

Proposition 2. *Let $A \in \mathcal{I}(n, n)$, $i, j \in N$, let r be a natural number. Then the entries of A^r are described by the formula*

$$a^r_{i,j} = \bigoplus \{w_d(p);\ p \in SR(i, j, r) \cup WR(i, j, r)\} \ . \tag{3}$$

Proof. The assertion follows from Definition 1 and Proposition 1. Clearly, by the definition of the operation $\otimes_d$, the weight of every inadmissible path p is zero. If the set in equation (3) is empty, then the result of the operation $\bigoplus$ is equal to zero, which corresponds to the fact that there are neither strong nor weak paths from i to j in this case, while the inadmissible paths have zero weights. □

Proposition 3.

$$w_d(p) > 0 \quad \text{if and only if} \quad p \in SR(i,j,r) \cup WR(i,j,r) \tag{4}$$

$$a^r_{i,j} > 0 \quad \text{if and only if} \quad \exists p \in SR(i,j,r) \cup WR(i,j,r) \tag{5}$$

Proof. The assertion follows directly from previous considerations. □

The last proposition in this section describes the conditions under which the weight of a path in max-drast is equal to 1 or to some value u with $0 < u < 1$.

Proposition 4. *For weight of a path p between nodes i and j of length r the following assertions hold true*

$$w_d(p) = 1 \quad \text{if and only if} \quad p \in SR(i,j,r) \tag{6}$$

$$0 < w_d(p) < 1 \quad \text{if and only if} \quad p \in WR(i,j,r) \tag{7}$$

Proof. This proposition results from the definition of strong and weak path. Weights of all edges in strong path are 1, hence the weight of such path is 1. Similarly, the weight of the weak path (with exactly one weak edge and remaining strong edges) is the least element $u \in (0,1)$. □

In the next section we shall consider subdigraphs of $G(A)$, called *threshold digraphs*, which will be denoted as $G(A,h)$, where $h \in \langle 0,1 \rangle$ is a threshold level. In the threshold digraph $G(A,h)$, such edges are only included, whose weight is equal or greater than level h. We also shall consider strongly connected components of $G(A,h)$. By standard definition, strongly connected component K is such a subset of nodes in the digraph that any two nodes i, j are contained in a common cycle.

Component K is called non-trivial if there is at least one cycle of positive length in K. The set of all non-trivial strongly connected components of digraph $G(A,h)$ will be denoted by $\mathrm{SCC}^*(G(A,h))$. For $K \in \mathrm{SCC}^*(G(A,h))$, the component period $\mathrm{per}(K)$ is defined as the greatest common divisor of the length of all cycles in K.

4 Computation of the matrix period

The max-min and max-drast algebra differ in the interpretation. A max-min matrix describes the flow capacities, while a max-drast matrix concerns the transitions reliability in the system.

Similarly as in the max-min algebra, by max-drast operations no new elements (except 0) are created. As a consequence, the matrices in the power sequence of matrix A in max-drast algebra only contain the entries from A.

The above mentioned reliability of the path is represented by powers of the matrix A. We can investigate paths of length that is even much longer than number of edges in the digraph of A. This means to compute high powers of matrix A. As a consequence of repetition of elements and possibility of creation only the zero element in the matrix, the period in the power sequence of A, $\mathrm{per}(A)$, must sooner or later occur.

Computation of the length of the matrix period in max-min algebra has been described in [7]. The matrix period in max-min algebra is computed by the following formula.

$$\mathrm{per}_{\min}(A) = \mathrm{lcm}\big\{\mathrm{per}(K);\ K \in \mathrm{SCC}^*(G(A,h)), h \in \langle 0,1 \rangle\big\}\ . \tag{8}$$

In computation of the matrix period in max-drast algebra, only the level $h = 1$ must be considered.

Theorem 5. *Let $A \in \mathcal{I}(n,n)$. Then*

$$\mathrm{per}_{\mathrm{drast}}(A) = \mathrm{lcm}\big\{\mathrm{per}(K);\ K \in \mathrm{SCC}^*(G(A,1))\big\}\ . \tag{9}$$

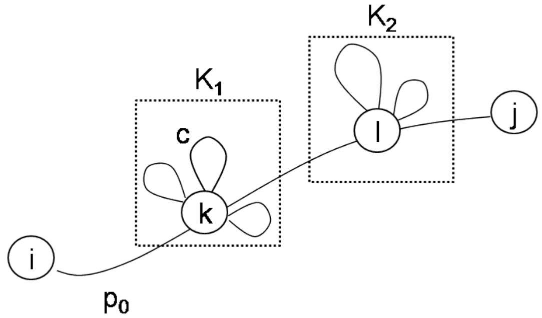

Figure 1: The path p_0

Proof. Let us denote $d^* = \operatorname{lcm} \operatorname{per}(K)$, where $K \in \mathrm{SCC}^*(G(A,1))$. We also denote the period of the matrix A in max-drast algebra as $d_A = \operatorname{per}_{\text{drast}}(A)$. It is the least d such that $\exists R, \forall r > R : A^r = A^{r+d}$. Determine the period of the elements of the matrix powers as $d_{i,j} = \operatorname{per}_{\text{drast}}(A_{i,j})$. This is also the least d such that $\exists R, \forall r > R : a^r_{i,j} = a^{r+d}_{i,j}$. Let us denote $d_A = \operatorname{lcm}\{d_{ij}, i,j \in N\}$. In this notation the assertion of the theorem can be simply written as $d_A = d^*$.

The proof will be done in two steps. In step (i) we prove that $d_A|d^*$, and in step (ii) that $d^*|d_A$.

(i) It holds that $\operatorname{lcm} d_{i,j}|d^*$ if and only if $(\forall i,j) : d_{i,j}|d^*$ - it is clear, that if least common multiple (e.g. $\operatorname{lcm} = a \cdot b$) divides some number, then also particular component of the multiple (e.g. a) divides the number. Consider non-trivial $K \in \mathrm{SCC}^*(G(A,1))$ and $p_0 \in SR(i,j,r) \cup WR(i,j,r)$ is a path of length r between i and j which goes through the strongly connected components K (see Figure 1).

Then we can find such $a^r_{i,j}(p_0)$ that equals to some $a^{r+d}_{i,j}(p_0)$. It holds that weight of such path is $w(p_0 + C) = w(p_0) \otimes_d w(C) = w(p_0)$, where C is certain combination of strong cycles in the component $C \subseteq K$. Because $d_{i,j}(p_0)$ is a period dependent on great common divisor of K, we can write that $d_{i,j}(p_0)|\operatorname{per}(K)|d^*$. Element of the r-th power of A is then $a^r_{i,j} = \max\{a^r_{i,j}(p_0)\}$. If $d_{i,j}(p_0)|d^*$ then also $d_{i,j}|d^*$. Let us denote $d^*_{i,j} = \operatorname{lcm} d_{i,j}(p_0)$. Let d be such that $\forall p_0 : a^r_{i,j}(p_0) = a^{r+d}_{i,j}(p_0)$ then also $a^r_{i,j}(p_0) = a^{r+d^*_{i,j}}_{i,j}(p_0)$ and we can write also that $a^r_{i,j} = a^{r+d^*_{i,j}}_{i,j}$ As we proved, $d_{i,j}|d^*$. $d^*_{i,j} = \operatorname{lcm} d_{i,j}(p_0)$ and thus $d^*_{i,j}|d^*$. This results in $d_{i,j}|d^*_{i,j}|d^*$.

(ii) We have to prove that $d^*|d_A$, i.e. $\operatorname{lcm}\{\operatorname{per}(K), K \in \mathrm{SCC}^*(G(A,1))\}|d_A$. Let $K \in \mathrm{SCC}^*(G(A,1))$, $i \in [K]$. We claim, that $\operatorname{per}(K)|d_{i,i}$ which implies $\operatorname{per}(K)|d_A$, i.e. $d^*|d_A$.

Proof of the claim: Let d be such that $a^r_{i,i} = a^{r+d}_{i,i}$ for all $r > R$. Then take such r, that $\operatorname{per}(K)|r$, then also $\operatorname{per}(K)|r + d$ and its clear that also $\operatorname{per}(K)|d$. As $d_{i,i}$ is such d, that $a^r_{i,i} = a^{r+d}_{i,i}$, then we have $\operatorname{per}(K)|d_{i,i}$. And $d_A = \operatorname{lcm} d_{i,i}$, hence $\operatorname{per}(K)|d_{i,i}|d_A$. □

From the computation of the drast operation we can see, that zero is the result of the case where $\max(x,y) < 1$. This result is thus insensitive to the variability of the elements x and y, which are less than one.

Theorem 6. *Let $A \in \mathcal{I}(n,n)$. If all elements of the matrix A, denoted as $u \in (0,1)$, are replaced by one constant value $c \in (0,1)$, then the period of matrix power in max-drast will not be changed.*

Proof. Reliability of the path of r steps from node i to node j, $a^r_{i,j}$, is computed as maximum of paths weighted by drast operation. It holds, that the element $a^r_{i,j}$ is equal to value of the most reliable path of length r. For computation of d, the period of the element $a_{i,j}$, consider the $p_0 \in WR(i,j,r) \cup SR(i,j,r)$ which goes through the strongly connected components K (see Figure 1). If $p_0 \in WR(i,j,r)$, then weight of the path is changed from u to c without the change in the period of the element. On the other side, if the path $p_0 \in SR(i,j,r)$ then two possibilities can arise. The first is that for computation of the element $a^{r+1}_{i,j}$ is used strong cycle, then $a^r_{i,j} = a^{r+1}_{i,j} = 1$. The second is that for computation of the element $a^{r+1}_{i,j}$ is used weak cycle and $a^{r+1}_{i,j} = c$. In any case at least of one element $a^{r+l}_{i,j}, l < d$ will be equal to 1, therefore the period can not be changed. □

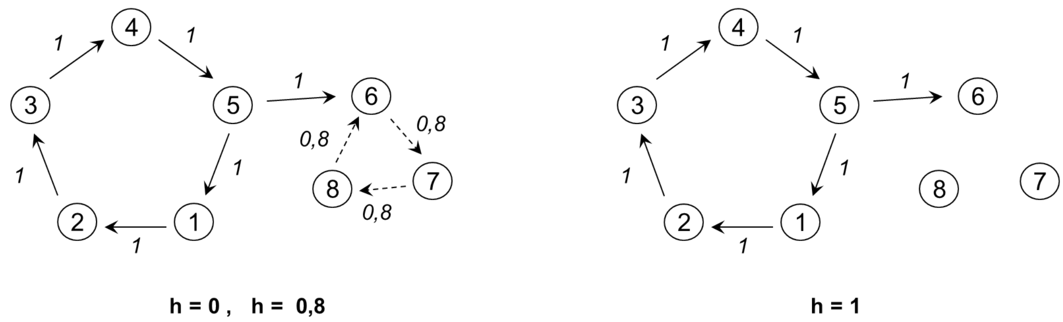

Figure 2: Treshold graphs

Theorem 7. *Let $A \in \mathcal{I}(n,n)$. Matrix power periods in max-min and max-drast equals if and only if edges on level $h \in u$ do not create new component C, such that*

$$\operatorname{lcm}\{\operatorname{per}(K);\ K \in \operatorname{SCC}^*(G(A,1))\} | \operatorname{per}(C) \ . \tag{10}$$

Proof. This results from the computation of the matrix powers periods in both algebras. In max-drast algebra only the level $h = 1$ must be considered. While in max-min algebra, periods of all non-trivial strongly connected components, K all levels can influence matrix power period by final computation of the least common multiple of $\operatorname{per}(K)$ through all levels. It is clear that by increase of the quantity of cycles in some component K_i, the greatest common divisor of this cycles can decrease. The values of periods differ in only one case, when on some level $h \neq 1$ appears such K_i, whose $\operatorname{per}(K_i)$ is not the divisor of the matrix period in max-drast, therefore this $\operatorname{per}(K_i)$ must be included into the computation of the least common multiple and will change the matrix power period in max-min. □

Theorem 8. *Let $A \in \mathcal{I}(n,n)$. Then it holds that*

$$\operatorname{per}_{\text{drast}}(A) | \operatorname{per}_{\min}(A) \ . \tag{11}$$

Proof. The calculation of matrix period in both algebras differs in final computation of least common multiple. Hence it holds, that the period in max-drast is a divisor of period in max-min. □

5 Example

Theorem 7 shows the case when period in max-min and max-drast equals. The following example (see the matrix A below and corresponding threshold graphs - Figure 2) represents the case when the period computed in both algebras differs.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0,8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0,8 \\ 0 & 0 & 0 & 0 & 0 & 0,8 & 0 & 0 \end{pmatrix} \tag{12}$$

Computation of the period in max-min algebra covers all threshold levels $h = \{0,8; 1\}$. At level $h = 0,8$ we can identify two non-trivial strongly connected components $K_1 = \{1,2,3,4,5\}$ and $K_2 = \{6,7,8\}$. Period of the first component, K_1, is equal to greatest common divisor of cycles in this component. Because the cycle is the only one, $\operatorname{per}(K_1) = 5$. Similarly, $\operatorname{per}(K_2) = 3$. At level $h = 1$ there is only one strongly connected component $K_3 = \{1,2,3,4,5\}$, $\operatorname{per}(K_3) = 5$. Then matrix power period in max-min algebra is computed as $\operatorname{per}_{\min}(A) = \operatorname{lcm}\{\operatorname{per}(K_1), \operatorname{per}(K_2), \operatorname{per}(K_3)\} = \operatorname{lcm}\{5,3,5\} = 15$.

In max-drast algebra only the $h = 1$ must be considered - there is exactly one non-trivial strongly connected component at this level, $K_4 = \{1, 2, 3, 4, 5\}$. Period of this component is also the greatest common divisor of cycles in the component: $\mathrm{per}(K_4) = 5$, then $\mathrm{per}_{\mathrm{drast}}(A) = \mathrm{lcm}\{\mathrm{per}(K_4)\} = \mathrm{lcm}\{5\} = 5$.

Acknowledgements

The support of INKOV project # CZ.1.07/2.3.00/20.0001 financed from EU and Czech Republic funds, and Czech Science Foundation project # 402/09/0405 is kindly acknowledged.

References

[1] Casasnovas, J., Mayor, G.: Discrete t-norms and operations on extended multisets. *Fuzzy Sets and Systems* **159** (2008), 1165–1177.

[2] Baccelli, F. L., and Cohen, G., and Olsder, G. J., and Quadrat, J. P.: *Synchronization and Linearity. An Algebra for Discrete Event Systems.* Wiley, Chichester, 1992.

[3] G. Cohen, D. Dubois, J. P. Quadrat, M. Viot, *A linerar-system-theoretic view of discrete event processes and its use for performance evaluation in manufacturing*, IEE Transactions on Automatic Control, AC-30 (1985), 210-220.

[4] R. A. Cuninghame-Green, *Describing industrial processes with interference and approximating their steady-state behavior*, Oper. Res. Quart. **13** (1962), 95-100.

[5] R.A. Cuninghame-Green: *Minimax Algebra*, Lecture Notes in Economics and Mathematical Systems **166**, Springer–Verlag, Berlin, 1979.

[6] R.A. Cuninghame-Green: *Minimax Algebra and Application*, in: Advances in Imaging and Electron Physics **90**, (P. W. Hawkes, Ed.). Academic Press, New York, 1995.

[7] Gavalec, M.: Computing matrix period in max-min algebra, *Discrete Appl. Math.*, **75** (1997), 63–72.

[8] Robertazzy, T.G.: *Planning Telecomunication Networks.* IEEE: Piscataway, NJ. 1999.

[9] Yi-Jia Tan, *On the powers of matrices over a distributive lattice*, Lin. Algebra Appl. **336** (2001), 1-14.

[10] U. Zimmermann: *Linear and Combinatorial Optimization in Ordered Algebraic Structure*, Ann. Discrete Math. **10**, North Holland, Amsterdam, 1981.