Interpretation of Dual Model for Piecewise Linear Programming Problem

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Abstract. Piecewise linear programming models are suitable tools for solving situations of non-linear nature. Several approaches for solving such models have been developed and are already known. Depending on type of the objective function (convex or non-convex), there are, among others, two common ways of solving these models. The solution of the models can either be found by plain modification of simplex algorithm or, if non-convex, in a different specific ways. Using an auxiliary model of integer (bivalent) programming is one way to deal with the problem. For each linear model, there is also dual model and its properties and purpose is widely known.

This paper is to describe dual associated model for general piecewise linear model with auxiliary model. This dual model provides an additional knowledge useful for solution of models that cannot be solved in any common way. The properties of this dual model would be described further in this paper. Dual model solution optimality and feasibility are discussed.

Keywords: Piecewise linear programming, duality, simplex algorithm.

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1 Introduction

Duality plays an important role in mathematical programming from both theoretical and computational point of view. [8] For those linear and convex problems very well known theory has been developed [3,12]. Studying duality for nonlinear programming problems has been of much interest in the past [7]. Various dual models have been developed. Dantzig et al. [4]. Mond [10] and Bazaraa and Goode [1] formulated a pair of symmetric dual models for nonlinear mixed integer programs. Mishra, Wang and Lai [9] followed that with their work, dealing with objective function convexity and concavity issues within such nonlinear models. Generally, more various specific models concerning duality of mathematical programming has been developed recently. Luc [8] dealt with duality in multiple objective linear programming, Fan [5] dealt with theories in nonlinear semidefinite programming and Mond and Weir [11] proposed a number of different duals for nonlinear programming problems with nonnegative variables and proved various duality theorems under appropriate pseudo-convexity/quasi-convexity assumptions.

There are many more approaches and models dealing with duality described by Mond and Chandra [2,10]. Difference between these duality models depends either on approach of defining and computing these models or on properties of original primal model. In this paper, let us focus on **piecewise linear model**. A general model of piecewise linear programming is defined in following way:

$$Z(x): \sum_{j=1}^{n} a_{ij}x_j \leq b_i, i = 1,2,\dots,m; x_j \geq 0, j = 1,2,\dots,n \tag{1}$$

Where $Z(x)$ is **objective function** consisting of number of fractional functions:

$$\begin{gathered} Z(x) = Z_1(x_1) + Z_2(x_2) + \cdots + Z_j(x_j) + \cdots + Z_n(x_n), \\ Z_j(x_j) = Z_{j1}(x_j), x_j \in \langle 0, k_{j1} \rangle \\ Z_j(x_j) = Z_{j1}(x_j), x_j \in \langle k_{j1}, k_{j2} \rangle \\ \vdots \\ Z_j(x_j) = Z_{jp_i}(x_j), x_j \in \langle k_{jp_i}, \infty \rangle, j = 1,2,\dots,n \end{gathered} \tag{2}$$

All of the functions Z_{jk} are considered to be linear.

The idea of this paper is to develop a dual model for specific primal model of nonlinear programming defined by Houška [6] called **general model of piecewise linear programming (PWLP)**. This particular model uses piecewise linear model with auxiliary model to solve problems of non-linear nature. Let us briefly describe this modified model and its components in the way Houška [6] suggested:

$$Z(x,r)\left|\begin{array}{c}\sum_{j=1}^{n} a_{ij}x_j \leq b_i, i = 1,2,\dots,m; x_j \geq 0, j = 1,2,\dots,n \\ x_j + r_{j(2k-1)} - r_{j(2k)} = k_{jk}, j = 1,2,\dots,n, k = 1,2,\dots,p_j \\ r_{jk} \geq 0, j = 1,2,\dots,n; k = 1,2,\dots,p_j \\ r_{j(2k-1)}.r_{j(2k)} = 0, j = 1,2,\dots,n; k = 1,2,\dots,p_j\end{array}\right| \quad (3)$$

Where the linear **objective function** is now following:

$$Z(x,r) = \sum_{j=1}^{n} c_j x_j + \sum_{k=1}^{p_j} c_{jk} r_{jk}, j = 1,2,\dots,n \quad (4)$$

Modified model obviously contains the **original constraints**:

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, i = 1,2,\dots,m$$

And additional **constraints of boundary points** with left distance variables $r_{j(2k-1)}$ and right distance variables $r_{j(2k)}$ regarding dividing points between subsequent linear functions in piecewise linear objective function:

$$x_j + r_{j(2k-1)} - r_{j(2k)} = k_{jk}, j = 1,2,\dots,n, k = 1,2,\dots,p_j \quad (5)$$

New **cost coefficients** are:

$$c_j \text{and } c_{jk} \quad (6)$$

Solution of the modified model still must remain **nonnegative**. This requires both constraints of original structure variables and constraints of distance variables:

$$x_j \geq 0, j = 1,2,\dots,n \quad (7)$$

$$r_{jk} \geq 0, j = 1,2,\dots,n; k = 1,2,\dots,p_j \quad (8)$$

Following **constraints containing nonlinear elements** can be omitted in the calculation itself, thus it has no effect on improving the value of the objective function:

$$r_{j(2k-1)}.r_{j(2k)} = 0, j = 1,2,\dots,n; k = 1,2,\dots,p_j \quad (9)$$

To use the standard simplex algorithm for solving this model, it is vital that the **objective function is convex** and following rules concerning cost coefficients have to be fulfilled:

For maximization model:

$$c_j \geq c_j + c_{j2} \geq c_j + c_{j2} + c_{j4} \geq \cdots \geq c_j + \sum_{k=1}^{p_i} c_{jk} \quad (10)$$

For minimization model:

$$c_j \leq c_j + c_{j2} \leq c_j + c_{j2} + c_{j4} \leq \cdots \leq c_j + \sum_{k=1}^{p_i} c_{jk} \quad (11)$$

For this particular model of piecewise linear programming with auxiliary model, the dual model and its properties will be described further in this paper.

2 General dual model

2.1 Primal model modification necessity

With this primal model given, it is possible to create appropriate general dual model of piecewise linear programming. The procedure of creating will be similar to transforming a simple linear model to dual model. A matrix of constraint coefficients must be transposed for the dual model. Cost coefficients of the primal model will make right-hand side coefficients of the dual model. Right-hand side coefficients of the primal model will make cost coefficients of the dual model. Equalities and inequalities of conditions in the primal model will affect (depending whether the objective functions is max or min) conditions of non-negativity of the dual model. Conditions of non-negativity in the primal model will affect equalities or inequalities in the dual model conditions. Maximization objective function will turn to minimization and vice versa. It is, however, necessary to deal with several difficulties that occur while transferring to dual model, while the transformation of general PWLP model cannot be done so easily like for the simple linear model.

The rudimentary problem appears that in the primal model, there are multiple objective functions. As the right-hand side vector of the dual model is strictly dependent on the coefficients of primal objective function, it is uncertain which of these values should that be, as there are more than one of each in the primal model piecewise linear objective function. The right-hand side values of the dual model actually determine the set of feasible solutions. Existence of number of linear functions in the primal objective function leads to very complicated set of feasibility solutions. The higher the number of piecewise linear functions in the primal, the greater number of conditions is to be met in the dual model. For example a primal model with a piecewise linear objective function consisting of two linear functions, each with two dividing points, leads to six more conditions to be met in the dual model. Generally the number of these conditions is determined by all combinations between subsequent linear functions in the primal objective function. The higher number of such conditions would make extremely large dual model difficult both for calculation and interpretation.

2.2 Dealing with the non-negativity of dual model

It is then wiser to transform a primal model to (3) first and create a dual model afterwards. The problem with the numerous coefficients of the primal objective function is not present anymore, as there is only one objective function and the right-hand side coefficients of the dual model can be determined easily. However the primal model is now larger in the terms of constraints of boundary points (5). These constraints will always exist in the primal model in the form of equalities. The form of restrictive conditions of the primal model directly affects non-negativity conditions (this term is slightly improper for the dual model, as these conditions don't have to be generally non-negative) of the dual model. Because of these constraints of boundary points in the primal model there will always be conditions *sine limitis* (the value of the variable can be whatsoever negative, non-negative or zero). Using the standard simplex method requires all of the variables to be non-negative. To meet this requirement it is necessary to transfer *s.l. (sine limitis)* conditions of dual model in the following way:

Original constraints of dual model $y_n = s.l.$ have to be replaced with these two conditions:

$$y_n \geq b_m; y_n \leq b_m \tag{12}$$

These conditions have the same meaning (there are no limitations of y values), however the condition of non-negativity still have not been met because of the condition $y_n \leq 0$. This condition can be easily turned non-negative by a simple substitution:

$$y_n = -y'_n \tag{13}$$

A new condition is created:

$$y'_n \geq 0 \tag{14}$$

This condition meets the requirement of non-negativity. It is of course necessary to substitute given variable wherever it appears in the whole dual model. This actually means multiplying it by -1.

2.3 Preserving feasibility of dual model

As it was described earlier the right-hand side coefficients in dual depend on cost coefficients of objective function in primal. The primal cost coefficient can be often negative. This is due the decreasing slopes of piecewise linear functions within the maximization objective function (10). Also the structure variables costs could be negative for various reasons depending on an interpretation of the problem. Should there be one or more negative right-hand side coefficients in the dual model, it is necessary to change them to positive by plain multiplying the whole equality (or inequality) by -1. The primal feasibility is vital for using a standard simplex method for solving any model.

2.4 Definition of the dual model

After dealing with the several problems regarding feasibility of the dual model, it is now possible to define a general dual model for primal piecewise linear model. This model can exist in two forms depending on whether the objective function is maximization or minimization. In the general model description, primal variable is only described generally as x (for both x and r, because the way of transforming to dual model is the same for both of them).

Maximization primal model

Pairs of appropriate parts will exist in primal and dual models as follows (m-rows, n-columns):

Matrices of conditions coefficients:

$$\begin{matrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2n}x_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}x_1 & a_{m2}x_2 & \cdots & a_{mn}x_n \end{matrix} \rightarrow \begin{matrix} a_{11}y_1 & a_{21}y_2 & \cdots & a_{m1}y_m \\ a_{12}y_1 & a_{22}y_2 & \cdots & a_{m2}y_m \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}y_1 & a_{2n}y_2 & \cdots & a_{nm}y_m \end{matrix} \tag{15}$$

Primal right-hand side vector and dual objective function:

$$\underset{b}{\rightarrow}(b_1, b_2, \dots, b_m)^T \quad \rightarrow \quad z(y) = b_1y_1 + b_2y_2 + \cdots b_my_m \rightarrow MIN \tag{16}$$

Primal cost coefficients and dual right-hand side vector:

$$z(x) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \rightarrow MAX \quad \rightarrow \quad \underset{c}{\rightarrow}(c_1, c_2, \dots, c_n)^T \tag{17}$$

Primal non-negativity conditions and dual conditions symbols:

$$x_n \geq 0 \quad \rightarrow \quad \sum a_{mn}y_m \geq c_n \tag{18}$$

Primal conditions symbols and dual non-negative conditions symbols:

$$\sum a_{mn}x_n \leq b_m \quad \rightarrow \quad y_m \geq 0 \tag{19}$$

$$\sum a_{mn}x_n \geq b_m \quad \rightarrow \quad y_m = -y'_m;\ y'_m \geq 0 \tag{20}$$

$$\begin{gathered} \sum a_{mn}x_n = b_m \Rightarrow \sum a_{mn}x_n \leq b_m \wedge \sum a_{(m+1)n}x_n \geq b_m \quad \rightarrow \quad y_m \geq 0; \\ y_{m+1} = -y'_{m+1};\ y'_{m+1} \geq 0 \end{gathered} \tag{21}$$

Minimization primal model

Pairs of appropriate parts will exist in primal and dual models as follows (m-rows, n-columns):

Matrices of conditions coefficients:

$$\begin{matrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2n}x_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}x_1 & a_{m2}x_2 & \cdots & a_{mn}x_n \end{matrix} \rightarrow \begin{matrix} a_{11}y_1 & a_{21}y_2 & \cdots & a_{m1}y_m \\ a_{12}y_1 & a_{22}y_2 & \cdots & a_{m2}y_m \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}y_1 & a_{2n}y_2 & \cdots & a_{nm}y_m \end{matrix} \tag{22}$$

Primal right-hand side vector and dual objective function:

$$\underset{b}{\rightarrow}(b_1, b_2, \dots, b_m)^T \quad \rightarrow \quad z(y) = b_1y_1 + b_2y_2 + \cdots b_my_m \rightarrow MAX \tag{23}$$

Primal cost coefficients and dual right-hand side vector:

$$z(x) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \rightarrow MIN \quad \rightarrow \underset{c}{\rightarrow}(c_1, c_2, \dots, c_n)^T \tag{24}$$

Primal non-negativity conditions and dual conditions symbols:

$$x_n \geq 0 \quad \rightarrow \quad \sum a_{mn}y_m \leq c_n \Rightarrow -\sum a_{mn}y_m \geq c_n \tag{25}$$

Primal conditions symbols and dual non-negative conditions symbols:

$$\sum a_{mn}x_n \geq b_m \quad \rightarrow \quad y_m \geq 0 \tag{26}$$

$$\sum a_{mn}x_n \leq b_m \quad \rightarrow \quad y_m = -y'_m;\; y'_m \geq 0 \tag{27}$$

$$\sum a_{mn}x_n = b_m \Rightarrow \sum a_{mn}x_n \geq b_m \wedge \sum a_{(m+1)n}x_n \leq b_m \quad \rightarrow \quad y_m \geq 0; \quad y_{m+1} = -y'_{m+1};\; y'_{m+1} \geq 0 \tag{28}$$

3 Illustrative example

Let us have following random **primal model of piecewise linear programming** containing 3 limiting conditions, non-negative conditions and piecewise linear function consisting of 6 linear sub-functions:

$$x_1 + x_2 \leq 24\,; \quad x_1 + 0{,}5x_2 \leq 16\,; \; x_1 \geq 10\,; \; x_{1,}x_2 \geq 0$$

$$z = Z(x_1) + Z(x_2) \rightarrow MAX$$

$$Z(x_1) = 1{,}5x_1 \; \in (0\,,2), Z(x_1) = x_1 + 1 \; \in (2\,,3), Z(x_1) = 0{,}5x_1 + 2{,}5 \; (3\,,\infty)$$

$$Z(x_2) = 2x_2 \; \in (0{,}1), Z(x_2) = x_2 + 1 \; \in (1\,,3{,}5), Z(x_2) = 0{,}6x_1 + 2{,}4 \; (3{,}5\,,\infty)$$

Transferring it to model (3) additional **boundary points conditions** are added:

$$x_1 + r_{11}^- - r_{11}^+ = 2\,; \; x_1 + r_{12}^- - r_{12}^+ = 3\,; \; x_2 + r_{21}^- - r_{21}^+ = 1\,; \; x_2 + r_{22}^- - r_{22}^+ = 3{,}5$$

And appropriate objective function is constructed:

$$z = 1{,}5x_1 - 2x_2 - 0{,}5r_{11}^+ - 0{,}5r_{12}^+ - r_{21}^+ - 0{,}4r_{22}^+ \rightarrow MAX$$

Using the standard simplex algorithm the optimal solution of this model can be found. Expressing it as a base solution vector the **solution is following**:

$$x_B^T = (10\,,12\,,0\,,8\,,0\,,7\,,0\,,11\,,0\,,8{,}5)$$

Optimal value of the objective function *z* is 17,1.

Let us find a dual model of the given primal model and find a solution of this dual model using a general model (15-28) described in the previous chapter. All necessary modifications preserving model feasibility and non-negativity has been made immediately in one step, resulting in the following model:

$$y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 \geq 1{,}5 \;\; ; \;\; y_1 + 0{,}5y_2 + y_8 - y_9 + y_{10} - y_{11} \geq 2$$

$$y_4 - y_5 \geq 0 \;\; ; \;\; y_4 - y_5 \leq 0{,}5 \;\; ; \;\; y_6 - y_7 \geq 0 \;\; ; \;\; y_6 - y_7 \leq 0{,}5$$

$$y_8 - y_9 \geq 0 \;\; ; \;\; y_8 - y_9 \leq 1 \;\; ; \;\; y_{10} - y_{11} \geq 0 \;\; ; \;\; y_{10} - y_{11} \leq 0{,}4$$

$$y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11} \geq 0$$

$$z(y) = 24y_1 + 16y_2 - 10y_3 + 2y_4 - 2y_5 + 3y_6 - 3y_7 + y_8 - y_9 + 3{,}5y_{10} - 3{,}5y_{11} \rightarrow MIN$$

If there is an existing optimal solution with the finite value of the objective function in the primal model, then there also has to be the same for the dual model. Using the standard simplex algorithm, the optimal solution of the dual model has been found. The **value of the objective function is again 17,1**. Because it is dual model the interpretation of itself is different from primal model. Optimal variable values of primal model can be found in the dual model as well, though they exist here as the shadow costs and in the optimal simplex table they can be found in optimality test row under the slack variables of dual model. Also their values are opposite (.-1), because it is necessary to have only negative values in the last row of the simplex table. Should the primal model be minimization and dual model maximization, then the optimal values in the dual simplex table would be the same as in primal simplex table. Generally, the appropriate pairs of values will be located in both simplex tables in the following way:

Primal model	**Dual model**
Optimal values of structure variables	Shadow costs of slack variables
Optimal values of slack variables	Shadow costs of structure variables
Dual costs of structure variables	Optimal values of slack variables
Dual costs of slack variables	Optimal values of structure variables

Table 1 Relations between primal and dual optimal solutions

4 Conclusion

The general dual model for primal piecewise linear programming model has been found and described. If one of these associated models have an optimal solution with a finite value of objective function then the second model also have an optimal solution with a finite value of objective function. Both values of objective function must be equal. Knowing optimal solutions of both associated problems, the dual values (dual costs) represent a change of the value of primary objective function upon unit increase of right-hand sides of conditions in primal problem. Optimal solution (optimal program) can exist if there is also an optimal valuation of factors. Dual model of piecewise linear programming makes it possible to do further interpretation of problems of non-linear nature, which can be primarily solved by primal model. Finding the solution of dual model brings other possibilities of analysis for given economical problem.

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