Convexity in stochastic programming model with indicators of ecological stability

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Abstract. We develop an optimization model dealing with construction expenses that are prescribed as a result of the EIA (Environmental Impact Assessment) process. The process is an obligatory part of every large construction project and evaluates possible influences of the project to the environment, including population health, natural and other socio-economic aspects; the result of the process is a set of recommendation and arrangements the construction must meet.

Our optimization model incorporates uncertainties in model parameters; we represent them through their probabilistic distribution. Furthermore, to overcome a problem with quantifying subjective utility function of ecological impacts, we measure them by so-called indicators of ecological stability. The resulting problem is stochastic programming problem formulated as (C)VaR model used traditionally in finance area. In our contribution we deal with convexity properties of this problem – these are especially important from the theoretical as well as from the computational point of view. We propose a series of assumptions to the problem that ensure convexity of the final set of feasible solutions and/or the objective function.

Keywords: EIA process, indicator of ecological stability, stochastic programming, value-at-risk models, convexity.

JEL classification: C44
AMS classification: 90C15

1 Introduction

In our previous contributions [6, 7] we started to analyze optimization models that deal with expenses connected with large, especially line constructions. We will continue in this direction and deal with convexity issues that are indispensably important from the theoretical and practical point of view.

1.1 Problem formulation

We recall here shortly the subject matter we deal with. During several last decades we notice a very considerable growth of number of small as well as big engineering constructions. This growth is accompanied by deeper regulations of the rules under which the constructions are rising. One of the most important changes compared to the past is the growing emphasis to the environment.

Every important construction must obey the so-called Environmental Impact Assessment (EIA) imposed by European Union Law (see [3]). This process is in fact evaluation of impacts of the construction to the environment and human healthy, divided into two classes and several categories:

- influence of the construction to the *human healthy*: noise pollution, air pollution and social-economic (comfort) factors (life conditions, transport services and loads, emotions);
- influence of the construction to to the *environment*: air, climate, water, land, forests, natural resources, flora, fauna, ecosystems, landscape, systems of ecological stability and also the tangible property or cultural heritage.

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The detailed list of the categories is given by [4] and the summary is also presented in [6]. The impacts are of very manifold nature – furthermore each of the factors involves many inputs and outputs and their evaluations is not simple. This all contribute to the fact that modelling the optimization problem is not easy and so is for calculation.

In the next section we recall the formulation of the mathematical model [7] and state some of its basic properties. We then focus on the convexity of the set of feasible solutions; we recall some known results from the literature that deals with convexity issues of chance-constrained probabilistic problems that can be more or less applied to our model.

2 Model description

The traditional cost-minimizing optimization model is not considered in this paper because of difficulties with non-linear utility function (representing the environmental impacts of the construction). Instead we consider model where costs are incorporated as the constraint with the right-hand side representing the budget limitation.

2.1 Uncertain optimization model

Denote $x \in X \subset \mathbb{R}^n$ the set of the possible compensations and arrangements ready to be used in some construction. The values of the variable x can be discrete or continuous according to the nature of the arrangement: they can be binary variables (to realize or not the arrangement), discrete (representing possible variants of the arrangements) as well as continuous (dimensions and other measures of the arrangement).

Let $\xi \in \Xi$ be the random vector representing uncertainty factors, with $\Xi \subset \mathbb{R}^s$ being the predefined support of ξ. The uncertainty factors represents traffic intensities (when building transport constructions), efficiency of compensating constructions, quantification of subjective criteria, accidents, etc. From the computational point of view, the probability distribution of ξ has to be known in advance.

The actual expenses of all arrangements are represented by the cost function $c : X \times \Xi \to \mathbb{R} : (x; \xi) \mapsto c(x; \xi)$. Suppose c to be a linear function of x, i. e., $c(x; \xi) = c^T x$ where $c \in \mathbb{R}^n$ are constant unit costs of the arrangements. This simplification is not crucial in our setting; possible dependence on random factor can be (later) incorporated in the probabilistic part of the model. Let furthermore B be the budget limit on the expenses.

The utility function $u : X \times \Xi : (x; \xi) \mapsto u(x; \xi)$ represents the factors of subjective and evaluative character. The quantification of this function is generally difficult. In our paper, we use the approach in which we replace the subjective utility function by a function based on the indicators of ecological stability.

We are now ready to write down the uncertain formulation of our model

$$\text{maximize } u(x; \xi) \text{ subject to } c^T x \leq B, x \in X_0, \tag{1}$$

where $X_0 \subset \mathbb{R}^n$ represents all deterministic constraints of the model (except costs).

2.2 Chance-constrained optimization model

We first deal with subjective utility function: as already outlined we replace it by indicators of ecological stability [4]. More precisely, let $i : \mathbb{R}^n \times \Xi \to \mathbb{R}^I : (x; \xi) \mapsto i(x; \xi)$ be a function representing the values of the EEA indicators, representing the functional dependence of the indicator value on decision vector x and random factor ξ. Next, we introduce a parameter L representing a required limit on values of i, and weights $w \in [0; 1]^I$ representing relative importance of each of indicators. The probabilistic information about ξ is incorporated model through the probabilistic constraint; we write the resulting final model in the form of

$$\text{maximize } L \text{ subject to } \mathbb{P}\{w^T i(x; \xi) \geq L\} \geq 1 - \varepsilon,\ c^T x \leq B, x \in X_0, \tag{2}$$

where $1 - \varepsilon$ ($\varepsilon \in [0; 1]$) represents some prescribed probability of fulfilling the required limit on indicators. The limit L is considered joint here and compensations for indicator values are allowed. Other approaches, such as treating the limits individually (or by groups of indicators), are possible but imply more unpleasant issues concerning convexity of the feasibility set. We will treat convexity of the set of feasible solutions in the next section.

3 Convexity analysis

3.1 Notion of generalized convexity

To analyze convexity properties of probability constraint of (2) we recall some results dealing with general forms of probabilistic constraints. Denote

$$X_\varepsilon := \big\{x \in X : \mathbb{P}\{g(x;\xi) \geq 0\} \geq 1-\varepsilon\big\} \tag{3}$$

the set with probabilistic constraint where $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^d$ is vector-valued mapping. If g is convex then the sets X_1, X_0 are convex. Unfortunately these sets are not of great interest: they represent existence and almost sure constraint fulfilling. The main "classical" result belongs to Prékopa [10] developed later especially by Borell [1] and [2]. To state the result we need introduce the notion of r-concave functions and measures.

Definition 1. A function $f : \mathbb{R}^d \to (0;+\infty)$ is called *r-concave* for some $r \in [-\infty;+\infty]$ if

$$f(\lambda x + (1-\lambda)y) \geq [\lambda f^r(x) + (1-\lambda) f^r(y)]^{1/r} \tag{4}$$

is valid for each $x, y \in \mathbb{R}^d$ and each $\lambda \in [0;1]$. The cases $r = -\infty, 0, +\infty$ are treated by continuity.

Among the r-concave functions we specify some important special cases which we enumerate in the following list:

- $r = -\infty$: the right hand side (RHS) of (4) is equal to $\min\{f(x), f(y)\}$ and f is called *quasi-concave* in this case. Quasi-concave functions play very important role in the context of probabilistic programming as we will see later;
- $r \in (-\infty; 0)$: f is r-concave function with negative r if f^r is convex function
- $r = 0$: RHS of (4) is equal to $f^\lambda(x) f^{1-\lambda}(y)$ and the function f is called logarithmically concave or log-concave (as $\log f$ is concave function). Log-concavity is useful property in stcohastic programming; many prominent probability densities share this property and the original Prékopa's results are formulated for log-concave functions (even that proofs are more general);
- $r = 1$: corresponds to classical notion of concavity;
- $r \in (0; +\infty)$: f is r-concave function with positive r if f^r is concave function;
- $r = +\infty$: the right hand side (RHS) of (4) is equal to $\max\{f(x), f(y)\}$ and f is called *quasi-convex* in this case.

If f is r^*-concave for some r^* then it is r-concave for all $r \leq r^*$. In particular, every r-concave function is also quasi-concave.

Definition 2. Probability measure $\mathbb{P}$ (on $\mathcal{B}(\mathbb{R}^s)$) is called *$r$-concave* if for any Borel convex set $A, B \subset \mathcal{B}(\mathbb{R}^s)$ such that $\mathbb{P}(A) > 0, \mathbb{P}(B) > 0$, and any $\lambda \in [0;1]$ we have

$$\mathbb{P}(\lambda A + (1-\lambda)B) \geq [\lambda \mathbb{P}^r(A) + (1-\lambda)\mathbb{P}^r(B)]^{1/r}$$

cases $r = -\infty, 0, +\infty$ are treated by continuity.

It is worth to note three important properties of r-concave probability measures:

- r-concave probability measure has r-concave distribution function;
- non-degenerate quasi-concave measure $\mathbb{P}$ on $\mathbb{R}^s$ with dimension s of the support has a density;

- r-concave density is equivalent to $\frac{r}{1+mr}$-concave measure on convex subset $\Omega \subset \mathbb{R}^s$ of dimension $m > 0$ (so for $r > -\frac{1}{m}$).

Proposition 1 ([9], Theorems 2.5 and 2.11)**.** *If $\mathbb{P}$ is absolutely continuous (with respect to Lebesgue measure), r-concave measure, and the one-dimensional components of g are quasi-concave functions of (x, ξ) then X_ε is convex set.*

The assumption on r-concavity distribution is equivalent to have γ-concave probability density with $\gamma = \frac{r}{1+rs} \geq -\frac{1}{s}$. Many distribution such that multivariate normal, uniform, Wishart, Beta, Dirichlet, Gamma and others have log-concave (or at least quasi-concave) distribution, or at least for some of theirs parameters (see e. g. [9] for details).

The above proposition plays a basement for more specific results. As r-concavity of probabilistic distribution is not a real obstacle, this is not true for quasi-concavity of function f. The main obstacle is that r-concavity is not preserved under addition for $r < 1$ nor under multiplication for $r < 0$. So more specific structures are examined in the literature in order to overcome these obstacles. We concentrate especially on situation corresponding to linear programming problems with random technology matrix; but the available results are still very rare in this case.

3.2 Problems with random technology matrix

The classical results are that of [8] and [13]:

Proposition 2. *If $\xi \in \mathbb{R}^s$ have non-degenerate multivariate normal distribution, b is constant scalar and $\varepsilon \leq \frac{1}{2}$ then the function*

$$G(x) = \mathbb{P}\{\xi^T x \leq b\}$$

is quasi-concave on $D = \{G(x) \geq \frac{1}{2}\}$, thus X_ε is convex.

The next result is due to [11] and generalizes previous theorem to the case of independent normally distributed rows of technology matrices.

Proposition 3. *If a random matrix T has independent normally distributed rows such that their covariance matrices are constant multiples of each other, and $\varepsilon \leq \frac{1}{2}$ then*

$$G(x) = \mathbb{P}\{Tx \leq b\}$$

is quasi-concave on $D = \{G(x) \geq \frac{1}{2}\}$ thus X_ε is convex.

Recently, [12] found equivalent condition on covariance matrices under the condition of uniform quasi-concavity of functions $G_i(x) := \mathbb{P}\{\xi_{i\cdot} \leq b_i\}$. Other interesting results are given by [5].

Proposition 4 ([5])**.** *Suppose T be the random matrix with pairwise independent normally distributed rows $\xi_{i\cdot}$ indexed by i having means μ_i and variance matrices Σ_i. Then X_ε is convex for*

$$\varepsilon < 1 - \Phi(\max\{\sqrt{3}, u^*\})$$

where Φ is standard normal distribution function,

$$u^* = \max_i 4\lambda_{\max}^{(i)} \left[\lambda_{\min}^{(i)}\right]^{-3/2} \|\mu_i\|,$$

and $\lambda_{\max}^{(i)}$, $\lambda_{\min}^{(i)}$ are the largest and the smallest eigenvalue of Σ_i.

3.3 Application to model with EEA indicators

Probabilistic constraint of the model (2) written in terms of (3) corresponds to the setting

$$g((x, L)^T; \xi) = w^T i(x; \xi) - L.$$

Function g is in general non-linear and verifying convexity become hard in that case: according to Proposition 1 one have to check if indicator function $i(x; \xi)$ is (jointly) quasi concave, i. e. to explore the structure of indicator function. If the indicator function depends linearly on x and ξ one can apply results of Section 3.2:

Theorem 5. *Suppose $i(x;\xi) = x^T A\xi$, random vector ξ having joint normal distribution; assume further $\varepsilon \le \frac{1}{2}$. Then the feasibility set of (2) is convex.*

Proof. Function $i(x;\xi)$ is normally distributed and depend linearly on x and ξ. In this case the assumption of Proposition 2 is fulfilled and the assertion of the theorem follows directly. □

Consider now the case where compensations (lack in one indicator can be superseded by surplus in another one) are prohibited. In this case we impose individual limit L_j for each indicator and the whole considered model becomes in fact the model with joint probability constraint:

$$\text{maximize } L \text{ subject to } \mathbb{P}\{i_j(x;\xi) \ge L_j,\ j \in \mathcal{J}\} \ge 1 - \varepsilon,\ c^T x \le B, x \in X_0, \tag{5}$$

where $\mathcal{J}$ is index set for individual indicator functions. Keep the (affine) linear dependence of ξ and x on indicator function. Nevertheless, we cannot use Kataoka's result (Proposition 2) as the model no longer possesses single constraint row. So the covariance structure of the problem enters into consideration: we have to check if assumption of one of Proposition 3 or 4 is fulfilled. The situation become much more worse without normality because no general results on convexity are not known at present time; one has to confide to approximation techniques in order to find (possibly) inner and outer convex approximation of the problem.

4 Conclusion

We have considered probabilistic formulation of a (non-financial) optimization problem which evaluates impacts of the activities (for example transport constructions) on the environment through so-called indicators of ecological activity. The problem is convex if assumption of normal distribution and affine linear dependence of indicators on decision vector and random factor is fulfilled. If this is not the case, no theoretical results are not known and only convex approximation of the original problem has to be considered.

Acknowledgements

This research was supported by the Czech Science Foundation under the project No. P402/10/0956.

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