Territory decomposition parameters of distribution tasks

Marta Janáčková[1], Alžbeta Szendreyová[2]

Abstract. Number of candidates for the location of centers and their locations affects the quality of the location problems solution. When the territory is evenly partitioned into the squares, the edge size impacts both the population density size in individual squares and numbers of qualitatively different squares on the border of the solved area. This paper deals with the relationship between the parameters.

Keywords: *p*-median, uncapacitated location problem, service center, private service systems, public service systems.

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1 Introduction

Both public and private service systems are involved in finding out a solution of many tasks. The private service systems take into account the maximization of the profit. On the other hand, the public service systems have to allow the approach to the common service for each customer (health services, education) in spite of disadvantageous conditions. The satisfaction of customer's demands is limited by many factors, such as financial resources, influence on the geographic environment, etc. The quality of the service can be evaluated by several criteria. The service systems properties are described in [3].

Uncapacitated location problem with limited number of service places is one of the service system tasks (*p*-median). The quality criterion of service provision can be the minimization of the maximal distance between a customer and its closest center. Another criterion is the minimization of the travel kilometers in accordance with the optimal average accessibility of service. The size of the set of candidates for the centers location impacts on the quality of the task result. The location of the candidates, the distribution of possible center locations [4], the transport infrastructure and other parameters impact the result as well. Some parameters mutually influence each other. Um Jaegon [2] deduced the relationship whereby the suitable number of candidates is in proportion to the population density in the region.

This paper deals with the possibilities how to achieve the similar status, if we solve the *p*-median problem by using our software. That means how to select the sets and candidates for possible centers that they evenly cover the serviced area and fulfill the specified proportional relationship.

2 Selection of candidates

Um Jaegon describes in [2] that the number of candidates for the service center location meets the expression $D=const.\rho^{2/3}$, where D is the density of candidates and ρ is the population density in the checked component. Our task is to place m candidates for the location. They should to fulfill the mentioned formula as good as possible.

For obtained set of candidates the *p*-median problem is solved, or the impact on the quality of service accessibility is evaluated. In this paper we consider that the quality of service is the total number of traveled kilometers.

We can select the set of candidates randomly, or the selection can be based on different criteria. In previous studies we found that for *p*-median we got better results, if the set contained more candidates. The results were better, if the candidates were placed in large cities as well. On the other hand, the necessity to ensure an equitable access of customers to the service makes us to deploy the candidates evenly throughout the served territory. Therefore, we divide the territory into smaller regular units and we choose the candidates for location from each group. The division of the territory to the individual components can be done in different ways. In some cases it is appropriate to maintain the compactness of the road network, or the borders of historic areas, respectively the division by regions. In the current proposal of the service system we use automated division of the area into squares.

The program for regular division of the area was created as a result of task solved in the thesis [1]. We used in our study its revised application. It allows to cover the serviced area with grid and to include the nodes of the

[1] University of Žilina, Univerzitná 1, 010 26 Žilina, Slovakia, Marta.Janackova@fstroj.uniza.sk.
[2] University of Žilina, Univerzitná 1, 010 26 Žilina, Slovakia, Alzbeta.Szendreyova@fri.uniza.sk.

network into each square of the grid. After the selection of the criteria, the program selects some network nodes and includes them into the set of candidates. The properties of the grid are determined by entering the side length of the square. It is possible to determine the square population and to calculate the density of the population.

It is clear that the side length of the square is in reverse proportion to the number of the squares. Since each square contains the same number of candidates, the number of squares is in direct proportion to the number of candidates. The side length of the square influences the assignment of nodes and hence the density of the population in each square. The question is how to choose the size of components, i.e. how to determine the side length of the square so that the population density in each square will not be very different. The side length of the square changes the boundaries of the components and also the assignment of the nodes to the components. The population density in the components (squares) depends on the side length of the square and on the geographical location of nodes as well. It is hard to pre-calculate how the density in the components will change. We handle the automated division of the area to the squares. It makes easy to change the number of the components and to process a large number of tasks. We use the obtained data as a basis for determining the number of candidates in each square according to the expression mentioned above.

The population densities in the squares of the grid are different due to geographical shape of the Slovakia territory. We expect that the density will align with increasing size of the square. Because each grid contains from tens to hundreds squares, we separate the squares into classes according to the population densities. We distribute the estimated number of candidates to the squares in proportion to the mentioned formula. A mean value from the interval of each class is used as the population density. The dependency on relationship between the number of candidates and the population density is not linear. The calculated values are not integer and we use them as a basis for determining the set of candidates. This set can be interactively adjusted. We compare the results and find out for which side length of the square will be the location of candidates according to the densities similar to their uniform distribution.

The results of the sorting and dividing a set of candidates into the squares are given in the chapter with the experiments.

3 *p*-median problem

To verify the properties of a decomposition of a candidates set, we solve the *p*-median problem. This is a known location problem with a limited number of possible center locations. Let I is the set of possible candidates for the center locations. Customers are placed at the network nodes J. Every municipality $j \in J$ has b_j customers. The segment between i and j is evaluated as a distance d_{ij} for each possible location $i \in I$ and each dwelling place $j \in J$. The *p*-median problem solves the location of p centers at some nodes from the set I. These p centers have to serve each customers $j \in J$ in order to minimize the number of traveled kilometers. We assume that all $b_j, j \in J$ customers will be served from exactly one center $i \in I$. In this case, the coefficient d_{ij} can be modified by product of distance d_{ij} and the number of customers b_j of the place $j \in J$. So, $c_{ij} = b_j . d_{ij}$ for $i \in I$ and $j \in J$.

The decision on locating or not locating a center i at a place $i \in I$ will be modeled by a variable y_i which takes the value of 1 if the center is located at place i and it takes the value of 0 otherwise. The decision on allocation of the customer from node j to the center at the place i is modeled by a variable z_{ij}. It takes the value of 1 if the customer j will be served from the center i and takes the value of 0 otherwise. A model of the problem follows:

$$\textit{Minimize} \quad \sum_{i \in I} \sum_{j \in J} c_{ij} z_{ij} \tag{1}$$

$$\textit{Subject to} \quad \sum_{i \in I} z_{ij} = 1 \qquad \text{for } j \in J \tag{2}$$

$$z_{ij} \leq y_i \qquad \text{for } i \in I,\ j \in J \tag{3}$$

$$\sum_{i \in I} y_i \leq p \tag{4}$$

$$y_i \in \{0,1\} \qquad \text{for } i \in I \tag{5}$$

$$z_{ij} \in \{0,1\} \qquad \text{for } i \in I,\ j \in J \tag{6}$$

The used coefficients have the following meanings:

c_{ij} … evaluation of the assignment;
p … required number of centers;
I … the set of possible service center locations;
J … the set of customers.

Expression (1) is the objective function. Constraints (2) guarantee that a customer from dwelling place $j \in J$ is assigned to the exactly one center $i \in I$. The constraints (3) ensure that the variable y_i takes the value of 1, when at least one customer is assigned to the center at place i. The constraint (4) put the limit p on the number of location.

4 Numerical experiments

The whole problem was solved on the road network of the Slovak Republic. The network consists of 2916 dwelling places with potential customers. The position of places and the numbers of inhabitants (potential customers) are known. The area of the Slovak Republic is about 49000 km^2.

The area was divided into squares on the basis of the square length. The length of squares acquires the values from 5 to 30 kilometers. A lot of squares did not include any dwelling place. In these squares no candidates were located, so it did not affect the result. Table 1 lists the number of squares according to the side length of the square in the grid.

a **[km]**	***V***	***O***	***O*** **[%]**	***S*** **[km^2]**
5	3440	1492	43.37	37300
10	860	516	60.00	51600
15	406	252	62.07	56700
20	220	152	69.09	60800
25	144	101	70.14	63125
30	105	73	69.52	65700

a-side length of the square, *V*-number of all squares, *O*-number of the inhabited squares, *S*-area of the inhabited squares

Table 1 Decomposition of the area

For better comparability we also present the percentage and the area of the inhabited squares. This area is not the same as the area of Slovak Republic because some of the squares are situated in the foreign territory and some squares are in inactive (without population) squares.

Squares, which were situated on the border of the area, had only a small number of dwelling places and there were also very small population in these squares. The densities in these squares were much smaller as the density of the squares with big cities. For each side length of the square we found the minimum and maximum densities of all active squares. We chose a smaller width of the class at the beginning of the interval for to separate the active squares with negligible density. Six classes for each grid were determined. In the classes were squares with the population density up to 1 percent of maximum density and further up to 5%, 25%, 50% and finally up to 100% of the maximum density of all the grid squares.

For each side length of the square, i.e. for each grid, we found the frequency of the classes, and we express the relative frequency with respect to the number of active squares of the grid. These data are shown in Table 2.

Density	**< 1%**	**1-5%**	**5-25%**	**25-50%**	**50-75%**	**<100%**
a=5; min=1.12; max=5528; number of squares=1493						
i	55.29	276.44	1382.22	2764.44	4146.66	5528.88
b_i	707	664	99	14	6	3
g_i	47.35%	44.47%	6.63%	0.94%	0.40%	0.20%
a=10; min=0.72; max=2740; number of squares=516						
i	27.40	137.01	685.07	1370.15	2055.22	2740.29
b_i	130	300	76	8	0	2
g_i	25.19%	58.14%	14.73%	1.55%	0.00%	0.39%

a=15; min=0.12; max=1423; number of squares=252						
i	14.23	71.17	355.87	711.74	1067.60	1423.47
b_i	31	123	89	7	1	1
g_i	12.30%	48.81%	35.32%	2.78%	0.40%	0.40%
a=20; min=0.53; max=695; number of squares=152						
i	6.95	34.75	173.77	347.55	521.32	695.09
b_i	10	36	89	13	2	2
g_i	6.58%	23.68%	58.55%	8.55%	1.32%	1.32%
a=25; min=1.13; max=700; number of squares=101						
i	7.00	34.98	174.90	349.79	524.69	699.58
b_i	10	16	67	6	1	1
g_i	9.90%	15.84%	66.34%	5.94%	0.99%	0.99%
a=30; min=0.20; max=445; number of squares=73						
i	4.45	22.25	111.23	222.46	333.69	444.92
b_i	3	6	46	15	2	1
g_i	4.11%	8.22%	63.01%	20.55%	2.74%	1.37%

i-upper limit of the interval *i* according to population density, b_i - number of squares in a class, g_i- percentage of b_i

Table 2 Classes according to the density

For each of decomposition, we created sets of candidates. They contained from 1 to 6 candidates in each inhabited square. The candidates were placed in communities with the largest population. We used the frequency of the classes and mean values of the densities to calculate the values (proportionality criteria) for each grid using the expression:

$$m_i = \frac{m.\rho_i^{2/3}}{\sum_{i=1}^{6} b_i.\rho_i^{2/3}} \tag{7}$$

where b_i is the frequency of the *i*-th class and ρ_i is the middle of the interval of *i*-th class. The closest integer value to m_i is a proposal to the number of the candidates for each square of *i*-th class. These values are listed in Table 3. The last value in Table 3 shows the percentage of candidates placed in the squares considering to the total number of candidates in the grid and according to this criterion.

Density	**< 1%**	**1-5%**	**5-25%**	**25-50%**	**50-75%**	**<100%**
a=5; min=1.12; max=5528; number of squares=1493						
m_i	0.34	1.13	3.32	6.11	8.58	10.74
h_i	16.26%	50.42%	21.98%	5.73%	3.45%	2.16%
a=10; min=0.72; max=2740; number of squares=516						
m_i	0.25	0.83	2.42	4.46	6.26	7.84
h_i	6.31%	48.11%	35.63%	6.91%	0.00%	3.04%
a=15; min=0.12; max=1423; number of squares=252						
m_i	0.17	0.56	1.65	3.03	4.26	5.34
h_i	2.10%	27.49%	58.17%	8.43%	1.69%	2.12%
a=20; min=0.53; max=695; number of squares=152						
m_i	0.11	0.38	1.10	2.03	2.85	3.57
h_i	0.75%	8.92%	64.51%	17.36%	3.75%	4.70%

a=25; min=1.13; max=700; number of squares=101						
m_i	0.12	0.38	1.12	2.06	2.89	3.62
h_i	1.15%	6.05%	74.12%	12.23%	2.86%	3.59%
a=30; min=0.20; max=445; number of squares=73						
m_i	0.09	0.30	0.86	1.59	2.24	2.80
h_i	0.37%	2.43%	54.49%	32.73%	6.14%	3.84%

m_i-corresponding number of candidates per square of a class, h_i - percentage of candidates pertaining to the class

Table 3 Proposed number of candidates according to population density

We solved the p-median task with p=100 for each decomposition, which was determined by the side length of the square and by sets of candidates. The objective function values depending on the side lengths of the square are presented in Table 4.

K \ *a*	*a*=5	*a*=10	*a*=15	*a*=20	*a*=25	*a*=30
1	31954028	33414347	34346567	36105439	45110994	---
2	38181714	33106788	32192284	33007281	34476130	36640763
3	48562293	33391232	31985927	32716026	33327815	32931305
4	---	34151736	31934509	31948971	33057442	32654166
5	---	36513845	32351365	31948971	32247703	32003904
6	---	40797022	33427365	31933226	32200492	31961588

a-side length of the square, *K*-number of candidates per square

Table 4 Values of the objective function

To solve the p-median task we used software product, which was implemented in the Department of Transport Networks, University of Žilina. It is based on the branch and bound method. Condition (3) of the model was ensured by Lagrange relaxation. Experiments were performed on a laptop with Windows Vista, 3.00 GB of RAM and 2.4 GHz processor.

5 Conclusions

The results show that with decreasing side length of the square, both the number of active squares and the maximum population density vary considerably. When the side length of the square is small (a=5), only a negligible amount of squares has population density greater than 5% of maximum. On the contrary, for a=30, only 12% of squares has a density less than 5% of maximum. In spite of these differences, the relative frequency with the square density of up to 25% of the maximum density reaches for all types of grid comparable values. While for a=5, a=10 it is 98%, for a=15 it is 96%, for a=20 reaches 89%, for a=25 to 92% and finally for a=30 reaches 75%.

Assignment the number of candidates according to the population density of the square is as follows: for a=5 proposal contains 0 to 10 candidates per square. When the side length a increasing the number of candidates decreases. For a=30 it is from 0 to a maximum of 3 candidates per square. If we sum up the squares in which the criterion recommends to place approximately 1-2 candidates (m_i is close to this value), then the amount for each of the decomposition exceeds 80% of the number of candidates.

Among the best results from the p-median problem for each of decompositions, there are no significant differences. If necessary, the result of automatic selection of a set of candidates according to the specified parameters can be adjusted manually.

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