A modification of Kaldor-Kalecki model and its analysis

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Abstract. This paper studies the investment cycle as an endogenous phenomenon using a modification of so called Kaldor-Kalecki's model as a tool of research. Our aim is to show that the (Kaldor-Kalecki) model which contains such features as non-linearity in the investment function, a delay between an investment decision and its delivery, has an ability to produce more complex behaviour of its variables, i.e. periodical or non-periodical oscillations. As the model is a system of two differential equations with delay which the common solvers are not able to reliably deal with so far, a special solver is developed by authors for this case and will be presented in the article.

Keywords: Kaldor-Kalecki model, system of non-linear differential equations, investment cycle, limit cycle, complex dynamics

JEL classification: C44
AMS classification: 90C15

1 Introduction

Economic cycles in traditional macroeconomics are closely connected with investment fluctuations. Examining thoroughly macroeconomic theories of endogenous cycle, we can find that two possible explanations of this phenomenon. It is the delay between an investment decision and the delivery of investment and the non-linearity in the investment function. The former approach can be found in the original paper from Kalecki [9], which later was also presented in Allen's work [1]. In his model, Kalecki assumes that there is a gestation period which is the time between the moment when an investment decision is made and the time of delivering of the finished real investment. The process of construction requires time and its mathematical description leads to a differential equation with delay which may exhibit more complex dynamics of model variables. In Allen's book, both the elder version of Kalecki's model (1935) and its later version proposed in (1943) and (1954) can be found and they are a generalisation of the original version. Both Kalecki's models describe the capital dynamics with the help of a linear differential equation with delay capable of generating periodical oscillation of capital.

The approach based on non-linearity of investment function usually uses a system of two or more differential equations. As we know, non-linear deterministic dynamic systems could display non-linear oscillations. The theory of non-linear deterministic systems can be found in Guckenheimers and Holmess famous work [5]. Their work attracted many followers, for example: Kuznetsov [12], Perko [14]. From the point of view of non-linear system theory, an economy is a non-linear system. This interpretation of economic systems is fundamental because according to it, the origin of economic fluctuations results from the inside structure of the system, not as a consequence of irregular external shocks as the real business cycles theory suggests.

Another approach to explain investment cycles can be found in Kaldor's original model in his seminal work [8]. While Kalecki's model is reduced to one differential equation with delay describing the capital formation, Kaldor's original idea is to study the evolution of production and capital formation. Kaldor suggests that the treatment of savings and investment as linear curves simply does not correspond to empirical reality. He assumed that investment and savings are both positive non-linear and non-convex functions of output (income) and that investment depends negatively on capital while savings dependence on capital is positive. From the non-convex shape of both curves and from their movement depending on capital he derived endogenous cyclical behaviour of production and capital.

Kaldor original approach has many modifications and improvements. Chang and Smyth [3] were the first

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authors who translated the original Kaldor's idea into a rigorous form. Modern forms of Kaldor's model and a detailed analysis of its solution could be found in works of Lorenz [13] or Gabisch and Lorenz [4]. Later, Kodera and Vosvrda [10] included Kaldor's model into a model of price dynamics and developed a model in which the dynamics of price level is strongly affected by oscillations of production and capital in the real sector.

To develop further the essential idea of endogenous business cycle theory, in our contribution we will connect both approaches (Kalecki and Kaldor) and modify the Kaldor-Kalecki model as a system of two differential equations with delay. Though this idea is not new thing and a high amount of papers devoting to this problem, for example, it can be found in Kaddar and Talibi Alaoui [7] or Krawiec and Szydlowski [11], our contribution will take a step further. Although we use the same general formulation of Kalecki-Kaldor's model as in the above mentioned papers, unlike these authors, we introduce a specific investment function of logistic form as one source of linearity. Moreover, without losing the general validity of results and conclusions, the savings function in our model is a linear function of logarithm of production. Above all, unlike other authors in mentioned papers we pay more attention to solving the Kalecki-Kaldor's model. Despite the fact that there are some toolboxes solving differential equations with delay, we do not find them as appropriate and reliable for this purpose, and therefore, we develop our own software tool for solving Kalecki-Kaldor's model as a system of delay differential equations.

The rest of our paper is structured as follows. In the next section, we are going to present the original Kaldor's model and show the cyclical behavior of solution for production and capital on a numerical example. In the third section the formulation of Kalecki-Kaldor's model is given. In fourth section, we will explain the method for numerical solution of Kalecki-Kaldor's model and apply it on a calibrated model. In the final section a short discussion as well as some features of obtained solution will be given.

2 Kaldor's Model

In this section, we will give a brief description of Kaldor's model with non-linear investment function. The model consists of two differential equations

$$\dot{Y}(t) = \alpha[I(Y(t), K(t)) - S(Y(t))], \tag{1}$$
$$\dot{K}(t) = I(Y(t), K(t)) - \delta K(t), \tag{2}$$

where $Y(t)$ denotes real production, $K(t)$ denotes capital, $I(Y(t), K(t))$ labels investment which is an increasing function of production and is decreasing in capital. The savings function is an increasing function of production and is denoted as $S(Y(t))$. The second equation describes capital formation which is given by the difference between investment I and capital consumption $\delta K(t)$ where δ denotes the rate of deprecation.

The investment function will be presented in a specific form. Let's leave aside for a moment that it is a function of time and assume it is a product of two functions $J(Y, K)$ and Y. Then $J(Y, K)$ is in fact an investment-product ratio frequently called the propensity to invest. It is common to assume that it is an increasing function of productivity of capital, therefore we have $J(Y, K) = J(Y/K)$. In our model, for convenience, we use the logarithmic form and the original function $J(Y, K)$ can be rewritten as $J(Y/K) = J(\mathrm{e}^{y-k}) = i(y - k)$, where $y = \log Y$, $k = \log K$. As the logistic function is chosen as the functional form for the propensity to invest $i(t)$, it can be defined as follows

$$i(y(t) - k(t)) = \frac{ai_0}{bi_0 + (a - bi_0)\mathrm{e}^{-a(y(t)-k(t))}}. \tag{3}$$

Though the choice of the propensity to invest function seems to be rather arbitrary and speculative, it meets all requirements of a function describing the propensity to invest. The graph of this function for parameters $a = 2$, $b = 8$, $i_0 = 1/8$ is shown on the Fig. 1. Further, it also has maximum lower bound and minimum upper bound (0 and a/b respectively. The investment function then is of the following form:

$$I(Y(t), K(t)) = \frac{ai_0}{bi_0 + (a - bi_0)\mathrm{e}^{-a(y(t)-k(t))}} Y(t). \tag{4}$$

By the same token, the real savings function is defined as a product of the propensity to save s and the production. We assume that the propensity to save is increasing in logarithm of production. This assumption shows that the propensity does not grow as quickly as production, but much slower. So the mathematical expression for the propensity to save function is

$$s(\log Y(t)) = s_0 + s_1 \log Y(t) = s_0 + s_1 y(t). \tag{5}$$

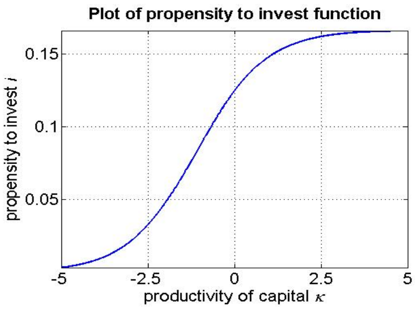

Figure 1 Logistic propensity to invest function

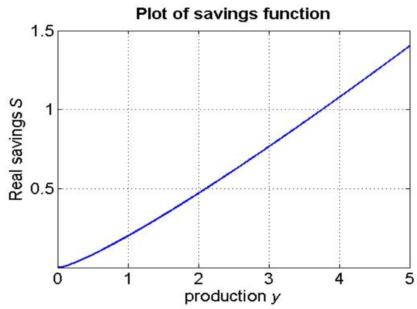

Figure 2 Real savings function

The savings function then is:

$$S(y(t)) = (s_0 + s_1 y(t))Y(t). \tag{6}$$

The real savings function with $s_0 = 0.2$ and $s_1 = 0.05$ is displayed in Fig. 2.

Using assumptions being made about investment and savings functions in (1) and (2), we get

$$\dot{Y}(t) = \alpha[i(y(t) - k(t))Y(t) - s(y(t))Y(t)], \tag{7}$$
$$\dot{K}(t) = i(y(t) - k(t))Y(t) - \delta K(t). \tag{8}$$

Dividing equation (7) by Y and (8) by K, we have

$$\dot{y}(t) = \alpha[i(y(t) - k(t)) - s(y(t))], \tag{9}$$
$$\dot{k}(t) = i(y(t) - k(t))\mathrm{e}^{y(t)-k(t)} - \delta, \tag{10}$$

where

$$\dot{y}(t) = \frac{\dot{Y}(t)}{Y(t)}, \qquad \dot{k}(t) = \frac{\dot{K}(t)}{K(t)}.$$

Substituting (3) and (5) into (9) and (10), we have

$$\dot{y}(t) = \alpha\left[\frac{ai_0}{bi_0 + (a - bi_0)\mathrm{e}^{-a(y(t)-k(t))}} - (s_0 + s_1 y(t))\right], \tag{11}$$
$$\dot{k}(t) = \frac{ai_0}{bi_0 + (a - bi_0)\mathrm{e}^{-a(y(t)-k(t))}} e^{y(t)-k(t)} - \delta. \tag{12}$$

The plot of the solutions of these two equations for $a = 2$, $b = 8$, $i_0 = 1/8$, $s_0 = 0.2$, $s_1 = 0.05$ and $\delta = 0.1$ is displayed on Fig. 3 and the phase portrait is shown on Fig. 4.

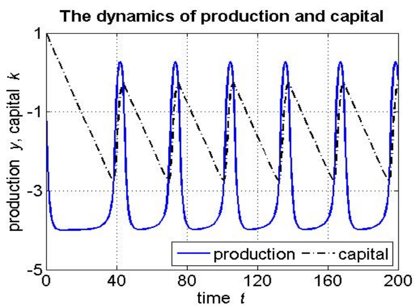

Figure 3 The evolution of production and capital

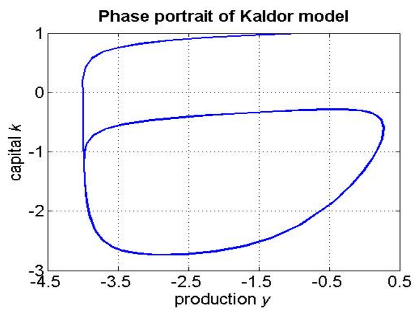

Figure 4 Phase portrait of Kaldor model

3 Kaldor-Kalecki model

Kalecki [9] in his famous article generalised Tinbergen's [15] idea of industrial investment cycles based on the ship-building model. Tinbergen observed gestation period from decision to invest to putting the investment to operation. He used a delay-differential equation for the description of this problem. Kalecki elaborated Tinbergen's original model using the idea of accelerator-multiplier joint activity. Accelerator-multiplier principle and the delay between investment decision and its delivery is expressed in delay-differential equation which models the process of capital formation. Tinbergen and Kalecki's idea of gestation period was utilized in modification of original Kaldor's model and resulted in Kaldor-Kalecki's model.

Kaldor-Kalecki's model takes into account a delay between investment decision and the delivery of investment. Capital formation denoted as $\dot{K}(t)$ is a difference between the investment decision performed in time $t-\theta$, $\theta > 0$ and capital consumption. Parameter θ stands for a gestation period. The first equation in Kaldor-Kalecki's model is the same as the one in original Kaldor's model. The decision to invest or the demand for investment is constituted in time t. The second equation is different because the investments delivered in time t are the result of investment decisions taken at time $t-\theta$. The Kaldor's system of differential equations (1) and (2) is modified to a new system as follows

$$\begin{aligned}\dot{Y}(t) &= \alpha[I(Y(t),K(t)) - S(Y(t))],\\ \dot{K}(t) &= I(Y(t-\theta),K(t-\theta)) - \delta K(t).\end{aligned}$$

As the assumptions about investment and savings functions remain unchanged the same, using

$$I(Y(t),K(t)) = i(y(t)-k(t))Y(t), \quad S(Y(t)) = s(y(t))Y(t),$$

we get the following system of differential equations

$$\begin{aligned}\dot{Y}(t) &= \alpha[i(y(t)-k(t))Y(t) - s(y(t))Y(t)],\\ \dot{K}(t) &= i(y(t-\theta)-k(t-\theta))Y(t-\theta) - \delta K(t).\end{aligned}$$

Dividing the first equation by $Y(t)$ and the second one by $K(t)$, we have

$$\begin{aligned}\dot{y}(t) &= \alpha[i(y(t)-k(t)) - s(y(t))],\\ \dot{k}(t) &= i(y(t-\theta)-k(t-\theta))\mathrm{e}^{y(t-\theta)-k(t)} - \delta.\end{aligned}$$

Using (3) and (5)and substituting them into the two differential equations above, we get

$$\dot{y}(t) = \alpha\left[\frac{ai_0}{bi_0 + (a-bi_0)\mathrm{e}^{-a(y(t)-k(t))}} - (s_0 + s_1 y(t))\right], \tag{13}$$

$$\dot{k}(t) = \frac{ai_0}{bi_0 + (a-bi_0)\mathrm{e}^{-a(y(t-\theta)-k(t-\theta))}} e^{y(t-\theta)-k(t)} - \delta. \tag{14}$$

Kaldor-Kalecki's model is a typical system of two delay differential equations. We will use the same set of parameters as in the case of original Kaldor's model to calibrate it.

4 Solving the model

Kaldor Kalecki's model as derived above is a system a two delay differential equations and in our paper we will solve it for a set of well chosen parameters. Unlike systems of ordinary differential equations which can be numerically solved relatively easily using the Runge-Kutta method, the presence of the lags in the right hand side makes systems of delay differential equations much more difficult to deal with. The difficulty of solving a delay differential equation is shown in the following example. Let's solve this simple DDE equation:

$$\dot{y}(t) = y(t-1). \tag{15}$$

Without the delay, this equation is an ODE equation of first order and can be solved easily both analytically and numerically [1]. We just need one initial condition to exactly identify the solution from a set of solutions. The first deviation from the ODE equation is that in order to identify the solution of (15), in stead of one initial condition, we need to know a whole series of initial conditions up to τ which is called history. Suppose that for $-1 \le t \le 0$ $y(t) = 1$. Then equation (15) becomes $\dot{y}(t) = 1$ with the initial condition $y(0) = 1$ and therefore for $0 \le t \le 1$, then the solution is $y = t + 1$. For $1 \le t \le 2$, then equation (15) becomes $\dot{y}(t) = t - 1 + 1 = t$ with the initial condition $y(1) = 2$ and the solution of 15 is $y = \frac{1}{2}t^2 + \frac{3}{2}$ and so on. Solving a delay differential equation results in solving a series of ODE equations within bounded intervals. As a result we get a solution with an important feature. The solution of equation (15) is continuous, but at $t = 0$ it has $\dot{y}(0_-) \neq \dot{y}(0_+)$, at $t = 1$ $\ddot{y}(1_-) \neq \ddot{y}(1_+)$ and generally at $t = k$ it has $y^{(k+1)}(k_-) \neq y^{(k+1)}(k_+)$. These points are discontinuous points and the solution

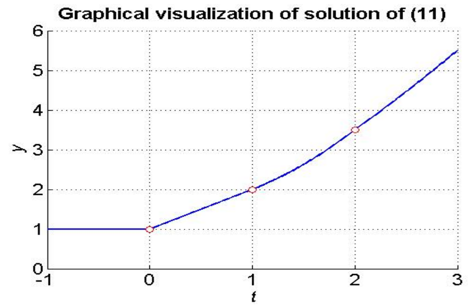

Figure 5 Solution of DDE equation (15)

of equation (15) is shown in Fig. 5. Since a DDE equation can be considered as a sequence of ODE equations, the Runge Kutta method can be used to numerically solve it. For an ODE equation of form $\dot{y}(t) = f(t, y(t))$, its numerical solution according to Runge and Kutta is as the following:

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i, \tag{16}$$

where h is the step length, b_i is the weight of k_i, s is the corresponding order of RK method and

$$\begin{aligned} k_1 &= hf(t_n, y_n) && (17)\\ k_i &= hf(t_n + c_i h, y_n + a_i k_{i-1}) \quad \text{for } i = 2, ..., s, && (18) \end{aligned}$$

where c_i and a_i are pre-set coefficients depending on the order s of the method.
In the case of a DDE equation of form $\dot{y}(t) = f(t, y(t), y(t-\tau))$ with history $y(t) = s(t)$ for $t \leq 0$, the Runge - Kutta method is modified to get the solution in the following way:

$$y_{n+1} = y_n + h \sum_{i=1}^{q} b_i k_i, \tag{19}$$

where

$$\begin{aligned} k_1 &= hf(t_n, y_n(t), y(t-\tau)) && (20)\\ k_i &= hf(t_n + c_i h, y_n + a_i k_{i-1}, y(t_n + c_i h - \tau)) \quad \text{for } i = 2, ..., s. && (21) \end{aligned}$$

Since for $t \geq 0$ the values of $y(t_n + c_i h - \tau)$ where $t_n + c_i h - \tau$ is the delay argument are unknown, we have to calculate them by interpolation. Due to the existence of discontinuities, when interpolating, the support points should be chosen in such way that they do not include discontinuities and the delay argument should be in the center of interval formed by those chosen support points. Numerical methods for solving ODE and DDE equations can be found in [2] and [6].
Using the method described above, Kaldor-Kalecki's system with the same values of parameters used for numerical calibration as in the case of original Kaldor model is numerically solved in the environment of MATLAB. The values of these parameters are $a = 2$, $b = 8$, $i_0 = 1/8$, $s_0 = 0.2$, $s_1 = 0.05$ and $\delta = 0.1$. The result is illustrated in Fig. 6 and Fig. 7. Solving this system, we realize that the dynamics of production and capital is very sensitive to the values of parameters as well as the delay and the system therefore can create a rich set of dynamics.

5 Conclusion

In our paper we have tried to revitalize the traditional Kaldor's and Kalecki's models, combine them together. Further, we introduce nonlinearity into the model in the form of logistic function. As such the model becomes a system of two delay differential equations with a constant delay. We also choose a set of parameters for the model and numerically solve it with Matlab. The solution of the system exhibits very rich dynamics with some chaotic feature. We consider the extension of our current work to two directions very interesting. First, relaxing assumption of one constant delay would make the model more realistic. Second, analyzing the data generated by the model and reconstructing the possible dynamics from real data would be an important step for verifying the validity of the model.

[1]Analytical solution of equation $\dot{y}(t) = y(t)$ is $y = \mathrm{e}^t$ for the initial condition $y(0) = 1$

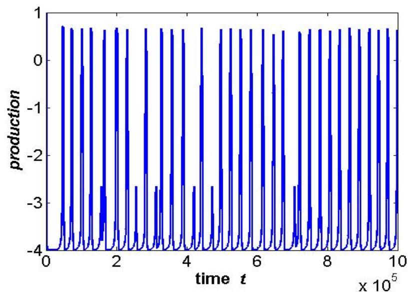

Figure 6 The dynamics of production

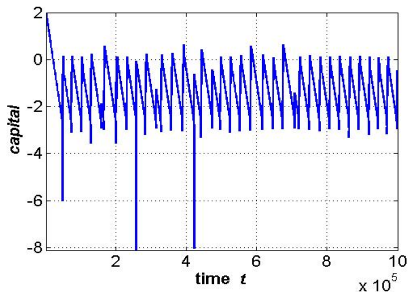

Figure 7 The dynamics of capital

Acknowledgements

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