Robustness and bootstrap approaches to SSD portfolio efficiency testing

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Abstract. This paper deals with portfolio efficiency testing with respect to second-order stochastic dominance (SSD) criteria. Unlike the pair-wise tests the portfolio efficiency tests allow for full diversification across the assets. As usual in SSD testing, the returns of assets are modeled by scenarios. We apply a computationally attractive method to test whether a US market portfolio, proxied by CRSP all share index, is SSD efficient with respect to 48 US industry representative portfolios. Moreover, we present a robust version of SSD portfolio efficiency test that allows for small errors in data and we analyze their impact on the market portfolio SSD efficiency. We enrich the results by the estimation of p-value of market portfolio SSD efficiency when the bootstrap technique to the data is applied.

Keywords: Portfolio efficiency, second-order stochastic dominance, robustness, bootstrapping

JEL classification: D81, G11
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1 Introduction

Portfolio efficiency tests with respect to second-order stochastic dominance (SSD) present a powerful non-parametric tool of decision-making theory for verifying the admissibility of a given portfolio for risk averse investors. The basics of decision-making theory were presented in the seminal work of Harry Markowitz [12]. He identified two main components of portfolio performance, mean reward and risk represented by variance. Applying a simple parametric optimization model he found the optimal trade-off between these two components. In this case, the portfolio is classified as efficient if there is no better portfolio, i.e., a portfolio with a higher mean and smaller variance. In the last 60 years, the theory of mean-risk models has been enriched by other risk measures, for example, semivariance, see [13], Value at Risk (VaR) or Conditional Value at Risk (CVaR), see [15], [17], [18].

More advanced application of risk measures in portfolio efficiency was introduced in Data Envelopment Analysis (DEA) models. Recently, Branda and Kopa [1] formulated DEA-risk models with risk measures as inputs and mean gross return as the output. These DEA-risk models can be seen as a generalization of mean-risk models, because they allow for multiple risk measure application. Moreover, if only one input is considered, then DEA-risk efficiency implies mean-risk efficiency with respect to the same risk measure. Branda and Kopa [1] compare the results of DEA-risk models with those of SSD portfolio efficiency.

Alternatively, one can adopt utility functions [14] for modelling investor's risk attitude, especially in the maximising expected utility approach. If the utility function is perfectly known, one can find the optimal decision. If that is not the case, one can at least identify the set of efficient portfolios with respect to a chosen class of utility functions. Considering all utility functions, that is, assuming only non-satiation for the investor's preferences, leads to the first-order stochastic dominance (FSD) relation (see [11] and references therein). Usually, we assume that the decision maker is risk averse, what reduces the considered class of utility functions. The admissible utility functions for risk averse investors are the concave ones. Therefore, adding the risk aversion assumption leads to the second-order stochastic dominance rules.

Using pairwise comparisons (e.g. [6]), an alternative (asset) is classified as SSD efficient if there is

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no other alternative that dominates the former alternative with respect to SSD. It means that a given portfolio is SSD pairwise inefficient if there exists some asset that is preferred by all decision makers. This is often very strong condition and even if it is violated still the alternative may not be the optimal choice for any investor. Therefore, Fishburn [5] defined the convex stochastic dominance efficiency as follows: an alternative is convex SSD inefficient (dominated by other alternatives) if every investor prefers at least one other alternative. Despite that, it does not cover the full diversification case if investors may combine assets in portfolios. Tests for SSD portfolio efficiency allowing full diversification across the assets were developed in [16], [10], and [8]. These tests classify a given portfolio as SSD portfolio efficient if there is no portfolio created from the assets that SSD dominates the portfolio. Alternatively, one can also apply FSD efficiency tests, see [10] and [9].

In all SSD efficiency tests a scenario approach to asset's returns is considered. Unfortunately, the results of these tests are very sensitive to changes in scenarios. Even a small perturbation of data matrix can completely change the SSD classification of a given portfolio, that is, a portfolio that was originally classified as SSD efficient turns to be SSD inefficient for perturbed data. Therefore, Kopa [7] suggests robust versions of SSD efficiency test based on δ-SSD portfolio efficiency. In these tests, small perturbations of the original scenarios are allowed and moreover, maximal possible changes in values of scenarios that do not change SSD classification are identified. Alternatively, one can adopt robustness approach of [4] that uses contamination techniques (recently applied also in [2]). Given the alternative distribution (for example stress scenario) on can explore the stability of SSD classification with respect to contaminated distributions. Dupačová and Kopa [4] introduced directional SSD portfolio efficiency approach that classifies portfolio as directionally SSD efficient (inefficient) if it is SSD efficient (inefficient) when using the original data as well as when the data are contaminated by an alternative distribution. They also derived conditions that are necessary or sufficient for directional SSD (in)efficiency.

Another way of dealing with high scenario sensitivity of SSD efficiency tests was proposed in [16] and [9]. They applied bootstrap techniques and evaluated the bootstrap p-value of market portfolio efficiency. They generated 10 000 pseudo-samples. In each pseudo-sample they tested portfolio efficiency of US market portfolio with respect to second-order [16] and first-order [9] stochastic dominance criteria. Their results reject FSD (SSD) portfolio efficiency with very high reliability.

In this paper we employ both robustness and bootstrap techniques and we compare the information obtained from results of both approaches. Contrary to [7], we introduce robust versions of the Post test instead of the modified Kuosmanen test. These new tests are more computationally attractive and allow for changes only in returns of market portfolio. Since returns of US market portfolio are proxied by (CRSP) all share index, we identify the maximal possible error in return scenarios of the market portfolio to preserve its SSD (in)efficiency. In bootstrap application, we follow [16] and [9] but we use new data set including during crises scenarios and we consider different representative (industry) portfolios as the basic assets.

The remainder of this paper is structured as follows. Section 2 presents notation, basic definitions and recalls SSD portfolio efficiency tests. It is followed by introduction of new tests, that can be seen as robust versions of the Post test. Section 4 shows the basic data description and presents the results of both tests as well as of the bootstrap approach. The paper is summarized and concluded in Section 5.

2 SSD portfolio efficiency tests

We consider a random vector $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ of returns of N assets with a discrete probability distribution described by T equiprobable scenarios. The returns of the assets for the various scenarios are given by

$$X = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^T \end{pmatrix}$$

where $\mathbf{x}^t = (x_1^t, x_2^t, \ldots, x_N^t)$ is the t-th row of matrix X representing the assets returns along t-th scenario. We assume that the decision maker may also combine the alternatives into a portfolio. We will use $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_N)^{\mathrm{T}}$ for a vector of portfolio weights and $X\boldsymbol{\lambda}$ represents returns of portfolio $\boldsymbol{\lambda}$. The portfolio possibilities are given by a simplex

$$\Lambda = \{\boldsymbol{\lambda} \in R^N | \mathbf{1}'\boldsymbol{\lambda} = 1, \ \lambda_j \geq 0, \ j = 1, 2, \ldots, N\},$$

which arises as the relevant case if we exclude short sales and impose a budget restriction. Moreover, the random return of the tested asset $\boldsymbol{\tau}$ is denoted by r_{N+1}. It takes values y^t, $t = 1, ..., T$ with equal probabilities. Without loss of generality, we assume that scenarios $\mathbf{x}^t$ are ascendantly ordered according to returns of the tested asset, that is, $y^1 \leq y^2 \leq ... \leq y^T$.

Following [11] and references therein, portfolio $\boldsymbol{\lambda}$ *dominates* asset $\boldsymbol{\tau}$ *with respect to second-order stochastic dominance* ($\boldsymbol{\lambda} \succ_{SSD} \boldsymbol{\tau}$) if $Eu(\mathbf{r}\boldsymbol{\lambda}) \geq Eu(r_{N+1})$ for all non-decreasing and concave utility functions with strict inequality for at least one such utility function. Alternatively, one can consider as a definition of this relation some of its necessary and sufficient conditions summarized in, for example, [7]. In any case, if portfolio $\boldsymbol{\lambda}$ dominates asset $\boldsymbol{\tau}$ with respect to second-order stochastic dominance then every risk averse decision maker prefers $\boldsymbol{\lambda}$ to $\boldsymbol{\tau}$ or is indifferent between them.

Following [16], we define the efficiency of a given portfolio with respect to second-order stochastic dominance relative to all portfolios that can be created from a considered set of assets.

Definition 1. A given asset $\boldsymbol{\tau}$ is *SSD efficient* if there exists a concave utility function u such that:

$$Eu(r_{N+1}) > Eu(\mathbf{r}\boldsymbol{\lambda}) \ \ \forall \boldsymbol{\lambda} \in \Lambda. \tag{1}$$

Otherwise, asset $\boldsymbol{\tau}$ is *SSD inefficient.*

Since Post [16] proved that the representative set of utility functions for SSD efficiency consists only of piece-wise linear concave functions, one can easily search for the admissible function (that is, function satisfying (1)) using linear programming techniques. Summarizing, if there is no tie in scenarios of tested asset, Post [16] derived the following SSD efficiency test.

Theorem 1. *Let*

$$\theta^* = \min_{\theta, \beta_t} \theta \tag{2}$$

$$\begin{aligned}
s.t. \qquad \sum_{t=1}^{T} \beta_t (y^t - x_n^t) + T\theta &\geq 0 \qquad n = 1, 2, ..., N \\
\beta_t - \beta_{t+1} &\geq 0 \qquad t = 1, 2, ..., T-1 \\
\beta_t &\geq 0 \qquad t = 1, 2, ..., T-1 \\
\beta_T &= 1.
\end{aligned}$$

A given asset $\boldsymbol{\tau}$ is SSD efficient if and only if $\theta^* \leq 0$.

If the tested asset is SSD efficient then an admissible utility function can be constructed from marginal utility levels β_t^* identified by (2) as the optimal solutions. On the other hand, the optimal coefficients β_t^* have no economic meaning for SSD inefficient portfolio.

Beside the Post test, one can alternatively use the Kuosmanen test [10] or the Kopa and Chovanec test [8] for SSD efficiency of a given asset or portfolio. Both these tests are more computationally demanding than the Post test, however they provide a useful information about a SSD dominating portfolio if the tested one is SSD inefficient.

The optimal value of the Post test (as well as of the Kuosmanen test and the Kopa and Chovanec test) can be seen as a degree of inefficiency. However, if the tested portfolio is SSD efficient these tests give no additional information. Therefore, Kopa [7] introduced a measure of SSD efficiency that is applicable for SSD efficient portfolios.

3 Robust versions of the Post SSD efficiency test

Since SSD efficiency of a given asset is very sensitive to any changes in scenario vector $\mathbf{y}$, we present robust versions of the Post test, that classify a given asset as SSD (in)efficient for the original and also for the slightly changed scenario vector. Contrary to [7], the scenario matrix X is assumed to be fixed. Let $\bar{\mathbf{y}} = (\bar{y}^1, ..., \bar{y}^T)$ be a perturbed scenario vector of returns of the tested asset. We consider the following distance between the original scenario vector and perturbed one:

$$d(\mathbf{y}, \bar{\mathbf{y}}) = \max_{1 \le t \le T} (\bar{y}^t - y^t)$$

.

For a given $\epsilon > 0$, we say that a tested asset is ϵ-SSD efficient if it is classified as SSD efficient for original scenario vector $\mathbf{y}$ as well as for all perturbed scenario vectors $\bar{\mathbf{y}}$ satisfying $d(\mathbf{y}, \bar{\mathbf{y}}) \le \epsilon$. Similarly, a tested portfolio is called ϵ-SSD inefficient if it is classified as SSD inefficient in the original case as well as in the case of perturbed scenarios that are from the ϵ-neighbourhood of $\mathbf{y}$.

Modifying Theorem 1, we can easily formulate a necessary and sufficient condition for ϵ-SSD efficiency of a given asset.

Theorem 2. *Let $z^t = y^t - \epsilon$, $t = 1, ..., T$ and*

$$\theta_E^* = \min_{\theta, \beta_t} \theta \tag{3}$$

$$\begin{array}{llcll} s.t. & \sum_{t=1}^{T} \beta_t (z^t - x_n^t) + T\theta & \ge & 0 & n = 1, 2, ..., N \\ & \beta_t - \beta_{t+1} & \ge & 0 & t = 1, 2, ..., T-1 \\ & \beta_t & \ge & 0 & t = 1, 2, ..., T-1 \\ & \beta_T & = & 1 & \end{array}$$

A given asset $\boldsymbol{\tau}$ is ϵ-SSD efficient if and only if $\theta_E^* \le 0$.

Proof. The choice $z^t = y^t - \epsilon$, $t = 1, ..., T$ represents the worst case of returns from ϵ-neighbourhood of $\mathbf{y}$. Therefore, if $\theta_E^* \le 0$ for $z^t = y^t - \epsilon$, $t = 1, ..., T$ then $\theta_E^* \le 0$ for any $\bar{\mathbf{y}}$ from ϵ-neighbourhood of $\mathbf{y}$ and hence the tested asset is ϵ-SSD efficient. On the other hand, if $\theta_E^* > 0$ for $z^t = y^t - \epsilon$, $t = 1, ..., T$ then $\bar{\mathbf{y}} = \mathbf{y} - \epsilon\mathbf{1}$ is the perturbation causing that the tested asset is not ϵ-SSD efficient. □

The robust test from Theorem 2 is again based on solving linear program. Hence, it is easy to solve using any linear programming solver (algorithm).

If the tested asset is SSD inefficient, then it is not ϵ-SSD efficient for any $\epsilon > 0$. The test of ϵ-SSD inefficiency can be easily derived from the Post test as follows.

Theorem 3. *Let $z^t = y^t + \epsilon$, $t = 1, ..., T$ and*

$$\theta_I^* = \min_{\theta, \beta_t} \theta \tag{4}$$

$$\begin{array}{llcll} s.t. & \sum_{t=1}^{T} \beta_t (z^t - x_n^t) + T\theta & \ge & 0 & n = 1, 2, ..., N \\ & \beta_t - \beta_{t+1} & \ge & 0 & t = 1, 2, ..., T-1 \\ & \beta_t & \ge & 0 & t = 1, 2, ..., T-1 \\ & \beta_T & = & 1 & \end{array}$$

A given asset $\boldsymbol{\tau}$ is ϵ-SSD inefficient if and only if $\theta_I^* > 0$.

The proof of Theorem 3 is very similar to the proof of Theorem 2, where $z^t = y^t + \epsilon$, $t = 1, ..., T$ is the best choice from ϵ-neighbourhood of $\mathbf{y}$.

4 Empirical study

To present the SSD efficiency tests, we apply them to historical US stock market data. We consider monthly excess returns from January 1982 to December 2011 ($T = 360$ observations) of $N = 48$ representative industry stock portfolios that serve as the base assets. The values are considered in percentage representation. The industry portfolios are based on four-digit SIC codes and they are from Kenneth French data library. We test whether a US market portfolio, proxied by CRSP all share index is SSD and ϵ-SSD (in)efficient relative to all portfolios that can be created from the considered 48 representative US industry stock portfolios. Moreover, if the tie in scenarios of market portfolio returns occurs we slightly modify the tied scenarios (adding very small value) to have no ties. It is needed mainly in bootstrap application. This adjustments have no impact on the efficiency testing results.

Firstly, we apply Theorem 1 to test SSD portfolio efficiency of the market portfolio. We find $\theta^* = 1.0432$ and hence the market portfolio is classified as SSD inefficient. The value of θ^* shows that the minimal value (over all concave utility functions) of maximal violation (over all 48 base assets) of efficiency criteria is equal to 1.0432%. Therefore, the maximal ϵ for which the market portfolio is ϵ-SSD inefficient is $\epsilon = 1.0432$. We can easily check it also using Theorem 3. It means that if the returns of market portfolio are proxied with accuracy smaller or equal to 1.0432, that is, the differences between theoretical values and proxied values of all return's scenarios are smaller or equal to 1.0432, then the market portfolio is always classified as SSD inefficient.

Finally we construct 10 000 pseudo-samples of the same length as the original data and we do the bootstrap testing with replacement. More details about various resampling techniques can be found in [3]. We find that the market portfolio is very strongly SSD inefficient. The inefficiency is detected in all 10 000 pseudo-samples and the estimated (bootstrap) p-value of SSD efficiency of US market portfolio is smaller than 0.0001.

5 Conclusions

This paper deals with robustness and bootstrap techniques in SSD portfolio efficiency testing. It presents new ϵ-SSD portfolio (in)efficiency tests as a robust generalization of the Post test. These tests are very computationally attractive requiring only solving linear programming problems.

In empirical application, we analyze SSD efficiency of US stock market portfolio relative to all portfolios that can be created from 48 representative industry portfolios. We find that the market portfolio is SSD inefficient. Moreover, this classification remains the same if we slightly perturb the returns of market portfolio. To reach SSD efficiency of market portfolio, one has to increase each its scenario by at least 1.0432. And bootstrap techniques show that this minimal required increase is very high, because the market portfolio is classified as SSD inefficient in all 10 000 pseudo-samples.

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