Characterization of uniformly quasi-concave functions

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Abstract. Quasi-concave functions appears in economics and finance as utility functions, measures of risk or other objects, mainly in portfolio selection analysis. A special attention was paid to these functions in the minimax theory. Unfortunately, their limited application is due to the fact that supremum, sum, product of quasi-concave functions are typically not quasi-concave. This difficulty is removed by establishing of uniformly quasi-concave functions, due to Prékopa, Yoda and Subasi (2011). Supremum and sum of uniformly quasi-concave functions are also a quasi-concave function. Moreover, product of nonnegative uniformly quasi-concave functions is a quasi-concave function. We contribute with a new characterization of uniformly quasi-concave functions that allows for easier verification and provide more straightforward insight. Hence, application and usage of uniformly quasi-concave functions become to be easier and more natural.

Keywords: Quasi-concave function, uniformly quasi-concave functions, partial ordering, total ordering, monotonicity.

JEL classification: C44
AMS classification: 90C15

1 Introduction

Let start with a definition of quasi-concave functions.

Definition 1. Let $E \subset \mathbb{R}^n$. We say that $f : E \to \mathbb{R}$ is quasi-concave if

1. E is convex.
2. For each $\alpha \in \mathbb{R}$ the level set $\mathsf{lev}_{\geq\alpha} f = \{x \in E : f(x) \geq \alpha\}$ is convex.

Alternatively, one can deal with quasi-convex functions. Function f is quasi-convex if and only if function $-f$ is quasi-concave. All these functions are useful in economics and finance, they serve as utility functions, measures of risk or other objects, mainly in portfolio selection analysis. In this paper we focus on quasi-concave functions. Unfortunately, their limited application is due to the fact that supremum, sum, product of quasi-concave functions are typically not quasi-concave. This difficulty is removed by establishing of uniformly quasi-concave functions, due to Prékopa, Yoda and Subasi (2011).

Definition 2. Let $E \subset \mathbb{R}^n$. Then, we say that functions $f_i : E \to \mathbb{R}$, $i = 1, 2, \ldots, m$ are uniformly quasi-concave if

1. E is convex.
2. For each $i = 1, 2, \ldots, m$ the function f_i is quasi-concave.
3. For each $x, y \in E$ either

$$\forall i = 1, 2, \ldots, m \quad \min\{f_i(x), f_i(y)\} = f_i(x)$$

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or

$$\forall i = 1, 2, \dots, m \quad \min\{f_i(x), f_i(y)\} = f_i(y).$$

We present an equivalent descriptions of uniformly quasi-concave functions.

2 Equivalent characterizations of uniformly quasi-concave functions

Any set of functions determines a partial ordering on their common domain. This observation allows us to show announced characterizations.

Let us consider a subset $E \subset \mathbb{R}^n$ and a finite family of functions $\mathcal{F} = \{f_i, i = 1, 2, \dots, m\} \subset \mathbb{R}^E$.

Definition 3. We say that functions of $\mathcal{F}$ are uniformly monotone if for each $x, y \in E$ either

$$\forall i = 1, 2, \dots, m \quad \min\{f_i(x), f_i(y)\} = f_i(x)$$

or

$$\forall i = 1, 2, \dots, m \quad \min\{f_i(x), f_i(y)\} = f_i(y).$$

We see that the functions of $\mathcal{F}$ are uniformly quasi-concave iff the functions of $\mathcal{F}$ are uniformly monotone and each of them is quasi-concave.

Definition 4. The set of functions $\mathcal{F}$ determines a partial ordering $\prec^{\mathcal{F}}$ and equivalence $\sim^{\mathcal{F}}$ on E by

$$x \prec^{\mathcal{F}} y \iff \begin{array}{l} \forall i = 1, 2, \dots, m : f_i(x) \leq f_i(y) \\ \exists j \text{ s.t. } f_j(x) < f_j(y) \end{array}$$

$$x \sim^{\mathcal{F}} y \iff \forall i = 1, 2, \dots, m : f_i(x) = f_i(y)$$

The partial ordering is giving an equivalent description of uniform monotonicity.

Theorem 1. *Functions of $\mathcal{F}$ are uniformly monotone iff the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}}/_{\sim^{\mathcal{F}}}$, i.e. for each couple $x, y \in E$ just one from the three following relations holds*

$$x \prec^{\mathcal{F}} y,\ x \sim^{\mathcal{F}} y,\ y \prec^{\mathcal{F}} x.$$

Proof. We will prove the equivalence.

1. Let functions of $\mathcal{F}$ are uniformly monotone.
   Fix $x, y \in E$, $x \not\sim^{\mathcal{F}} y$.
   Then, there is an index j s.t. $f_j(x) \neq f_j(y)$.
   We have to distinguish two possibilities:
   (a) Let $f_j(x) < f_j(y)$.
   Hence from uniform monotonicity $\forall i = 1, 2, \dots, m : f_i(x) \leq f_i(y)$.
   Consequently, $x \prec^{\mathcal{F}} y$.
   (b) Let $f_j(x) > f_j(y)$.
   Hence from uniform monotonicity $\forall i = 1, 2, \dots, m : f_i(x) \geq f_i(y)$.
   Consequently, $y \prec^{\mathcal{F}} x$.

   We have proved that the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}}/_{\sim^{\mathcal{F}}}$.
2. Let the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}}/_{\sim^{\mathcal{F}}}$.
   Fix $x, y \in E$.
   Since the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}}/_{\sim^{\mathcal{F}}}$, we have to distinguish three possibilities:

(a) If $x \sim^{\mathcal{F}} y$ then, $\forall i = 1, 2, \dots, m$ $\min\{f_i(x), f_i(y)\} = f_i(x) = f_i(y)$.
(b) If $x \prec^{\mathcal{F}} y$ then, $\forall i = 1, 2, \dots, m$ $\min\{f_i(x), f_i(y)\} = f_i(x)$.
(c) If $y \prec^{\mathcal{F}} x$ then, $\forall i = 1, 2, \dots, m$ $\min\{f_i(x), f_i(y)\} = f_i(y)$.

We have shown that functions of $\mathcal{F}$ are uniformly monotone.

□

Characterization by partial ordering implies characterization by composition of appropriate functions.

Theorem 2. *The following statements are equivalent:*

1. *Functions of $\mathcal{F}$ are uniformly monotone.*
2. *There are a function $\psi : E \to \mathbb{R}$ and non-decreasing functions $\varphi_i : \psi(E) \to \mathbb{R}$, $i = 1, 2, \dots, m$ such that $f_i = \varphi_i \circ \psi$ for all $i = 1, 2, \dots, m$.*
3. *There are a function $\psi : E \to \mathbb{R}$ and non-decreasing functions $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \dots, m$ such that $f_i = \varphi_i \circ \psi$ for all $i = 1, 2, \dots, m$.*

Proof. To prove announced equivalences, we prove step by step chain of implications.

1. Evidently, (3) ⇒ (2) and (2) ⇒ (1).
2. Let functions of $\mathcal{F}$ are uniformly monotone.
   We set $\psi = f_1 + f_2 + \dots + f_m$.
   Hence,
   $$\begin{aligned} x \prec^{\mathcal{F}} y &\iff \psi(x) < \psi(y), \\ x \sim^{\mathcal{F}} y &\iff \psi(x) = \psi(y), \\ y \prec^{\mathcal{F}} x &\iff \psi(x) > \psi(y). \end{aligned}$$
   For $i = 1, 2, \dots, m$: we define $\varphi_i : \psi(E) \to \mathbb{R}$ such that for $d \in \psi(E)$ we set
   $$\varphi_i(d) = f_i(x) \iff \psi(x) = d.$$
   The definition is correct because of
   $$\psi(y) = d \iff x \sim^{\mathcal{F}} y \iff \forall i = 1, 2, \dots, m : \; f_i(x) = f_i(y).$$
   (a) We have constructed a function $\psi : E \to \mathbb{R}$ and non-decreasing functions $\varphi_i : \psi(E) \to \mathbb{R}$, $i = 1, 2, \dots, m$ such that $f_i = \varphi_i \circ \psi$ for all $i = 1, 2, \dots, m$. Thus, statement (2) is fulfilled.
   (b) For each $i = 1, 2, \dots, m$, we extend the function φ_i to the whole $\mathbb{R}$ by
   $$\begin{aligned} \xi_i(t) &= \sup\{\varphi_i(d) : d \le t, \; d \in \psi(E)\} \quad \text{if} \quad \exists d \in \psi(E) \text{ s.t. } d \le t, \\ &= \inf\{\varphi_i(d) : d \in \psi(E)\} \quad \text{if} \quad \forall d \in \psi(E) : \; d > t. \end{aligned}$$
   The function ξ_i is non-decreasing and for all $d \in \psi(E)$ we have $\xi_i(d) = \varphi_i(d)$.
   The final tasks is to show that ξ_i does not reach infinite values.
   i. Assume $t \in \mathbb{R}$ such that $\xi_i(t) = -\infty$.
   Then, there is a sequence $d_n \in \psi(E)$, $n \in \mathbb{N}$ such that $d_n > t$ for all $n \in \mathbb{N}$ and $\varphi_i(d_n)$ is tending to $-\infty$.
   Consequently for any $x_n \in E$ such that $\psi(x_n) = d_n$, we observe $f_i(x_n) = \varphi_i(d_n)$ is tending to $-\infty$.
   According to definition of ψ and uniform monotonicity
   $\psi(x_n) = f_1(x_n) + f_1(x_n) + \dots + f_1(m) = d_n$ is tending to $-\infty$.
   That is a contradiction since we assume $t \in \mathbb{R}$ with $d_n > t$ for all $n \in \mathbb{N}$.

ii. Assume $t \in \mathbb{R}$ such that $\xi_i(t) = +\infty$.
Then, there is a sequence $d_n \in \psi(E)$, $n \in \mathbb{N}$ such that $d_n \leq t$ for all $n \in \mathbb{N}$ and $\varphi_i(d_n)$ is tending to $+\infty$.
Consequently for any $x_n \in E$ such that $\psi(x_n) = d_n$, we observe $f_i(x_n) = \varphi_i(d_n)$ is tending to $+\infty$.
According to definition of ψ and uniform monotonicity
$\psi(x_n) = f_1(x_n) + f_1(x_n) + \cdots + f_1(m) = d_n$ is tending to $+\infty$.
That is a contradiction since we assume $t \in \mathbb{R}$ with $d_n \leq t$ for all $n \in \mathbb{N}$.

We have constructed a function $\psi : E \to \mathbb{R}$ and non-decreasing functions $\xi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, m$ such that $f_i = \xi_i \circ \psi$ for all $i = 1, 2, \ldots, m$. Thus, statement (3) is fulfilled.

□

The observations can be summarized to give equivalent description for uniformly quasi-concave functions.

Theorem 3. *The following statements are equivalent:*

1. *Functions of $\mathcal{F}$ are uniformly quasi-concave.*
2. *Functions of $\mathcal{F}$ are uniformly monotone and each of them is quasi-concave.*
3. *Each function of $\mathcal{F}$ is quasi-concave and the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}} /_{\sim^{\mathcal{F}}}$.*
4. *There are a quasi-concave function $\psi : E \to \mathbb{R}$ and non-decreasing functions $\varphi_i : \psi(E) \to \mathbb{R}$, $i = 1, 2, \ldots, m$ such that $f_i = \varphi_i \circ \psi$ for all $i = 1, 2, \ldots, m$.*
5. *There are a quasi-concave function $\psi : E \to \mathbb{R}$ and non-decreasing functions $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, m$ such that $f_i = \varphi_i \circ \psi$ for all $i = 1, 2, \ldots, m$.*

Proof. The proof is based on the observation that the functions of $\mathcal{F}$ are uniformly quasi-concave iff the functions of $\mathcal{F}$ are uniformly monotone and each of them is quasi-concave.

The other equivalent statements follow descriptions of uniform monotonicity derived in Theorem 1 and Theorem 2. Moreover, construction $\psi = f_1 + f_2 + \cdots + f_m$ in the proof of Theorem 2 is giving quasi-concave function ψ since sum of uniformly quasi-concave functions is quasi-concave, for proof see Prékopa, Yoda and Subasi (2011). □

3 Empirical examples

This section presents three examples of uniformly quasi-concave functions from decision making theory. These examples illustrate new characterization of quasi-concave functions in a simple way, where functions φ_i and ψ are easily identified. However, in general, the analytic prescription of φ_i and ψ may be much more demanding to find.

Example 1
Utility functions, introduced in von Neumann and Morgenstern (1944), are one of the basic tools of decision making theory. Especially, if a portfolio selection model maximizes the expected utility of the final wealth a proper choice of the particular utility function is very important. Depending on the investor's risk attitude we basically distinguish between three classes of utility functions: concave (suitable for risk averse investor), linear (risk neutral investor) and convex (risk seeking investor). In this example, we choose one utility function from each class, that is, $\mathcal{F} = \{u_1, u_2, u_3\}$ where $u_1(W) = e^W, u_2(W) = W, u_3(W) = -e^{-W}, \quad W \in \mathbb{R}$

Since all three utility functions are increasing, the uniform monotonicity of $\mathcal{F}$ can be easily verified. Moreover, each function of $\mathcal{F}$ is quasi-concave, because u_1 is an increasing convex function and u_2, u_3 are concave functions. Therefore we can apply Theorem 3 and find a simple characterization with quasi-concave inner function $\psi(W) = W$ and non-decreasing outer functions $\varphi_i(W) = u_i(W)$, $i = 1, 2, 3$.

Example 2
Following Pratt (1964) we may express the investor's risk attitude by (Arrow-Pratt) absolute risk aversion

measure (function) that is derived from twice differentiable utility function as the ratio of the second and the first derivative of the utility function:

$$r(W) = -\frac{u''(W)}{u'(W)}.$$

Since almost all investors are risk averse with decreasing absolute risk aversion measure we limit our attention on utility functions having Hyperbolic Absolute Risk Aversion (HARA functions):

$$r(W) = \frac{1}{aW+b}, \quad W \in \mathbb{R}^+$$

with $a, b > 0$. We add also the limiting cases when either $a = 0$ or $b = 0$. Summarizing, we again consider a family of three quasi-concave functions: $\mathcal{F} = \{r_1, r_2, r_3\}$ where

$$\begin{aligned} r_1(W) &= \frac{1}{aW+b}, \quad W \in \mathbb{R}^+, \quad a, b > 0 \\ r_2(W) &= \frac{1}{aW}, \quad W \in \mathbb{R}^+, \quad a > 0 \\ r_3(W) &= \frac{1}{b}, \quad W \in \mathbb{R}^+, \quad b > 0. \end{aligned}$$

The positive constants a, b guarantees uniform monotonicity of $\mathcal{F}$ because all functions in $\mathcal{F}$ are non-increasing. Moreover, all these functions are monotone and convex and hence quasi-concave. Summarizing, functions in $\mathcal{F}$ are uniformly quasi-concave. Using Theorem 3 we can represent them as follows $r_i = \varphi_i \circ \psi$, $i = 1, 2, 3$ where:

$$\begin{aligned} \psi(W) &= \frac{1}{W}, \quad W \in \mathbb{R}^+ \\ \varphi_1(x) &= \frac{x}{bx+a}, \quad x \in \mathbb{R}^+, a, b > 0 \\ \varphi_2(x) &= \frac{x}{a}, \quad x \in \mathbb{R}^+, a > 0 \\ \varphi_3(x) &= \frac{1}{b}, \quad b > 0. \end{aligned}$$

Of course the choice of functions φ_i, $i = 1, 2, 3$ and ψ is generally not unique and one can alternatively consider the following characterization:

$$\begin{aligned} \psi(W) &= \frac{b}{aW}, \quad W \in \mathbb{R}^+, a, b > 0 \\ \varphi_1(x) &= \frac{x}{xb+b}, \quad x \in \mathbb{R}^+, b > 0 \\ \varphi_2(x) &= \frac{x}{b}, \quad x \in \mathbb{R}^+, b > 0 \\ \varphi_3(x) &= \frac{1}{b}, \quad b > 0. \end{aligned}$$

In both cases the inner functions are quasi-concave and the outer functions are non-decreasing. Moreover, in the second case, the outer functions do not depend on a.

Example 3

Finally, we present the example of uniformly quasi-convex deviation measures. Deviation measures are derived from risk measures, see Rockafellar et al. (2006), that typically appear either in mean-risk portfolio selection models or in portfolio efficiency testing with respect to mean-risk criteria or stochastic dominance relation. See e.g. Branda and Kopa (2012), Kopa (2010) and references there in for more details. We consider normally distributed returns of M assets $\mathbf{r} \sim N(\mathbf{m}, V)$. Investor may combine these assets into portfolios with weights $\mathbf{w} \in \Lambda$. We do not allow short positions and impose the budget constraint, that is,

$$\Lambda = \{\mathbf{w} \in \mathbb{R}^M | \mathbf{1}'\mathbf{w} = 1, \quad w_i \geq 0, \quad i = 1, 2, ..., M\}.$$

We consider conditional Value at Risk at level $\alpha \in [0, 1]$ ($CVaR_\alpha$), introduced in Rockafellar and Uryasev (2000, 2002), as the suitable risk measure. For portfolio $\mathbf{w}$, it can be computed as follows:

$$CVaR_\alpha(Z) = \min_{\eta \in \mathbb{R}} \left\{ \eta + \frac{1}{1-\alpha} \mathbb{E}[Z - \eta]^+ \right\},$$

where $[\cdot]^+ = \max\{\cdot, 0\}$ denotes the positive part, $Z = -\mathbf{r}'\mathbf{w}$ and η is a real auxiliary variable. Since we assume normally distributed returns the formulation of conditional Value at Risk of portfolio $\mathbf{w}$ can be simplified to:

$$CVaR_\alpha(-\mathbf{r}'\mathbf{w}) = -\mathbf{m}'\mathbf{w} + g(\alpha)\sqrt{\mathbf{w}'V\mathbf{w}}$$

where $g : [0, 1] \to \mathbb{R}^+$ is increasing function, see Rockafellar and Uryasev (2000). Following Rockafellar et al. (2006) we construct the corresponding deviation measure ($DCVaR_\alpha$) as $CVaR$ of deviation between portfolio random loss and it expected value, that is:

$$DCVaR_\alpha(-\mathbf{r}'\mathbf{w}) = CVaR_\alpha(-\mathbf{r}'\mathbf{w} + \mathbf{m}'\mathbf{w}) = g(\alpha)\sqrt{\mathbf{w}'V\mathbf{w}}$$

The last equation follows from coherency of CVaR, see Artzner et. al (1999). As it is typical in risk shaping with CVaR (see Rockafellar and Uryasev (2002)) or stochastic dominance constraints (see e.g. Dentcheva and Ruszczynski (2006) or Kopa and Chovanec (2008)), we consider DCVaRs at particular levels α_k, $k = 1, 2, ..., K$. Each $DCVaR_{\alpha_k}(-\mathbf{r}'\mathbf{w})$ is quasi-convex because the variance matrix is positive-semidefinite. Since deviation measures are typically minimized and theory of quasi-concave function is formulated for maximization problems, we simply consider $f_k(\mathbf{w}) = -DCVaR_{\alpha_k} = -g(\alpha_k)\sqrt{\mathbf{w}'V\mathbf{w}}$, $k = 1, 2, ..., K$ that are quasi-concave functions. Moreover $g(\alpha_k)$ are always positive, and hence $\mathcal{F} = \{f_1(\mathbf{w}), ..., f_K(\mathbf{w})\}$ is uniformly monotone. Summarizing, $\mathcal{F}$ is uniformly quasi-concave and applying decomposition from Theorem 3, we can easily find quasi-concave inner function $\psi(\mathbf{w})$ and non-decreasing outer functions $\varphi_k(x)$:

$$\begin{aligned} \psi(\mathbf{w}) &= -\sqrt{\mathbf{w}'V\mathbf{w}}, \quad \mathbf{w} \in \Lambda \\ \varphi_k(x) &= g(\alpha_k)x, \quad x \in \mathbb{R}^+, k = 1, 2, ..., K. \end{aligned}$$

Acknowledgements

The research was supported by the Czech Science Foundation (grant P402/12/G097).

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