Multistage risk-averse asset allocation with transaction costs

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Abstract. This paper deals with asset allocation problems formulated as multistage stochastic programming models. Dynamic models allow rebalancing the portfolio multiple times before the final investment horizon is reached. The CVaR risk measure is used for its favorable properties and time consistent model is developed. The risk-aversion coefficients are separate for every stage, which covers a wide range of investment strategies. For example one could start with an aggressive investment policy, but focus on reducing the risk as the final stage approaches. The stock prices are assumed to be interstage independent and to follow lognormal distribution. The Stochastic Dual Dynamic Programming algorithm is then applied to solve the presented models. An extensive numerical study based on the data from Prague Stock Exchange compares the results obtained from static two stage models with the results from dynamic models having multiple stages. Both cases, with or without the transaction costs, are considered. The computational part of this work is realized in C++ language and uses CPLEX to solve the linear programs.

Keywords: Stochastic multistage programming, stochastic dual dynamic programming algorithm, Monte Carlo sampling, risk averse optimization.

JEL classification: C63, D81, G11
AMS classification: 49M25

1 Introduction

Mean-risk models are well-known instruments in portfolio selection analysis. These two-criteria models both maximize the portfolio mean return as well as minimize the risk which is linked to the stock market trading. Basics of portfolio selection theory using variance and semivariance as a measure of risk were published in the article [7] and the book [8] by Harry Markowitz already in 1950s. Since that time, many improvements have been proposed and implemented. There are new ways of measuring risk, let us mention at least Conditional Value at Risk [11] (CVaR), which will be used in our model. Moreover, several different distributions, for instance log-normal distribution, have been applied in mean-risk models.

The latest research is focused on dynamic models which allow rebalancing the portfolio multiple times before the final investment horizon is reached, we refer to the book [4] for an extensive overview of stochastic modeling in finance. In this paper, we present one of these models, asset allocation problem with transaction costs formulated as multistage stochastic program, inspired by the article [12]. Wide range of different investment strategies can be covered by the model, because risk-aversion settings can be adjusted separately for each stage.

We solve the model using stochastic dual dynamic programming algorithm (SDDP) which originated in the work of Pereira and Pinto [9]. SDDP-style algorithms rely on the assumption of stage-wise independence to provide good performance for problems with multiple stages. In order to apply the SDDP algorithm, a discrete scenario tree with finite number of nodes in each stage has to be built. This can be done from historical market data, by sampling from some continuous-type distribution or by sampling using the moment matching method, presented in [5]. New problems arise in the multistage setting, such as how many descendants should be used for each stage, how to deal with the scenario count explosion or how to estimate the distributions for every stage. From many articles dealing with these and associated problems we refer at least to [2], [3] and [10].

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We organize the remainder of this article as follows: We present our risk-averse multi-stage model in the next section and provide a brief description of the SDDP algorithm in Section 3. Computational results are presented in Section 4 and we conclude and discuss ideas for future work in Section 5.

2   Asset-allocation model

Presented model is inspired by the articles [12] and [14]. Evolution of the assets price in stages $t = 2, \ldots, T$ is supposed to be random, we denote $\boldsymbol{\xi}_t = \boldsymbol{p}_t$ as the ratio between the price of the assets in stage t and stage $t-1$. The first stage portfolio value is set to be equal to 1 and only the probability distributions governing future realizations $\boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_T$ are assumed known, independent of each other. The realization of $\boldsymbol{\xi}_2$ is known when decisions $\boldsymbol{x}_2$ must be made and so on up to the stage T. The decisions vectors $\boldsymbol{x}_t$ contain the optimal asset allocation strategy, while the term $\boldsymbol{p}_t \boldsymbol{x}_{t-1}$ captures the state of the system, meaning the total value of the assets in our portfolio. The history of the data process up to time t will be denoted $\boldsymbol{\xi}_{[t]}$, meaning $\boldsymbol{\xi}_{[t]} = (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_t)$.

Our model gives user the possibility to choose risk aversion coefficient and confidence level $\lambda_t, \alpha_t \in (0,1)$ separately for each stage of the model. In order to give a nested formulation of the model we will introduce following function which calculates expectation and risk coming from a random loss Z:

$$\rho_{t,\boldsymbol{\xi}_{[t-1]}}[Z] = (1 - \lambda_t)\,\mathbb{E}\left[Z|\boldsymbol{\xi}_{[t-1]}\right] + \lambda_t\,\mathrm{CVaR}_{\alpha_t}\left[Z|\boldsymbol{\xi}_{[t-1]}\right]. \tag{1}$$

We will suppose that short-selling of assets is not allowed. The risk-averse multistage asset-allocation model with T stages can be written in the following nested form:

$$\min_{\mathbf{1}^T \boldsymbol{x}_1 = 1, \boldsymbol{x}_1 \geq 0} \rho_{2,\boldsymbol{\xi}_{[1]}} \left[\min_{\boldsymbol{p}_2^T \boldsymbol{x}_1 = \mathbf{1}^T \boldsymbol{x}_2, \boldsymbol{x}_2 \geq 0} -\mathbf{1}^T \boldsymbol{x}_2 + \rho_{3,\boldsymbol{\xi}_{[2]}} \left[\min_{\boldsymbol{p}_3^T \boldsymbol{x}_2 = \mathbf{1}^T \boldsymbol{x}_3, \boldsymbol{x}_3 \geq 0} -\mathbf{1}^T \boldsymbol{x}_3 + \cdots \right.\right.$$
$$\left.\left. + \cdots + \rho_{T,\boldsymbol{\xi}_{[T-1]}} \left[\min_{\boldsymbol{p}_T^T \boldsymbol{x}_{T-1} = \mathbf{1}^T \boldsymbol{x}_T, \boldsymbol{x}_T \geq 0} -\mathbf{1}^T \boldsymbol{x}_T \right] \right] \right]. \tag{2}$$

Presented model differs from the other possible approaches by taking the risk measure as a function of the recourse value at each stage. It provides easy and explanatory policy construction rules and by nesting the optimal values we achieve the property of time consistency, which is also an important indicator about the model quality, see [15]. The interpretation of the objective function is not straightforward, but it can be viewed as the cost we would be willing to pay at the first stage instead of incurring the sequence of random costs $Z_1, \ldots, Z_T$, see [13].

Our model also allows dynamic programming equations to be developed, for the detailed description we refer again to the article [14]. Using the definition of Conditional Value at Risk:

$$\mathrm{CVaR}_\alpha[Z] = \min_u u + \frac{1}{\alpha}\mathbb{E}\left[Z - u\right]_+ \tag{3}$$

in (1) the recourse value $Q_t(\boldsymbol{x}_{t-1}, \boldsymbol{\xi}_t)$ at stage $t = 2, \ldots, T$ equals the optimal value of the problem:

$$\begin{aligned} \min_{\boldsymbol{x}_t, u_t} & -\mathbf{1}^T \boldsymbol{x}_t + \lambda_{t+1} u_t + \mathcal{Q}_{t+1}(\boldsymbol{x}_t, u_t) \\ \text{s. t. } & \boldsymbol{p}_t^T \boldsymbol{x}_{t-1} = \mathbf{1}^T \boldsymbol{x}_t \\ & \boldsymbol{x}_t \geq 0, \end{aligned} \tag{4}$$

with $\mathcal{Q}_{T+1}(\cdot) \equiv 0$ and recourse function calculated from the following equation:

$$\mathcal{Q}_{t+1}(\boldsymbol{x}_t, u_t) = \mathbb{E}\left[(1 - \lambda_{t+1})\,Q_{t+1}(\boldsymbol{x}_t, \boldsymbol{\xi}_{t+1}) + \frac{\lambda_{t+1}}{\alpha_{t+1}}\left[Q_{t+1}(\boldsymbol{x}_t, \boldsymbol{\xi}_{t+1}) - u_t\right]_+\right]. \tag{5}$$

Please note that in contrast with the risk-neutral multistage problem, additional decision variable u_t is introduced to represent the estimated Value at Risk level, see (3). The recourse function does not

depend on $\boldsymbol{\xi}_{[t]}$, because the process is assumed to be stage-wise independent. First stage optimal solution of the stochastic program is given by:

$$
\begin{aligned}
\min_{\boldsymbol{x}_1, u_1} \ & \lambda_2 u_1 + \mathcal{Q}_2(\boldsymbol{x}_1, u_1) \\
\text{s. t. } & \mathbf{1}^T \boldsymbol{x}_1 = 1 \\
& \boldsymbol{x}_1 \geq 0.
\end{aligned}
\tag{6}
$$

After the introduction of auxiliary variables u_t, the problem was converted to a case that resembles the risk-neutral model. However, there is still nonlinear function present in our equations (4)–(5), and the standard version of the SDDP algorithm cannot be directly applied.

We still have to extend our model to cover possible transaction costs. We will consider the case where transaction costs are proportional to the value of the assets sold or bought, f_{tr} denoting the relative cost. The balancing equation between stage $t-1$ and stage t portfolios has to be modified to include the total cost of $f_{tr}\mathbf{1}^T|\boldsymbol{x}_t - \boldsymbol{x}_{t-1}|$. Converting to linear formulation we replace the equations (4) with following dynamic programming equations $(t = 2, \ldots, T)$:

$$
\begin{aligned}
\min_{\boldsymbol{x}_t, u_t} \ & -\mathbf{1}^T \boldsymbol{x}_t + \lambda_{t+1} u_t + \mathcal{Q}_{t+1}(\boldsymbol{x}_t, u_t) \\
\text{s. t. } & \boldsymbol{p}_t^T \boldsymbol{x}_{t-1} - f_{tr}\mathbf{1}^T \boldsymbol{z}_t = \mathbf{1}^T \boldsymbol{x}_t \\
& \boldsymbol{z}_t \geq \boldsymbol{x}_t - \boldsymbol{x}_{t-1} \\
& \boldsymbol{z}_t \geq \boldsymbol{x}_{t-1} - \boldsymbol{x}_t \\
& \boldsymbol{x}_t \geq 0.
\end{aligned}
\tag{7}
$$

We suppose that all the initial capital is used for investment into stocks, meaning transaction costs are constant for the first stage and can be therefore omitted. The first-stage stochastic program remains in the following form:

$$
\begin{aligned}
\min_{\boldsymbol{x}_1, u_1} \ & \lambda_2 u_1 + \mathcal{Q}_2(\boldsymbol{x}_1, u_1) \\
\text{s. t. } & \mathbf{1}^T \boldsymbol{x}_1 = 1 \\
& \boldsymbol{x}_1 \geq 0.
\end{aligned}
\tag{8}
$$

3 Stochastic dual dynamic programming

We provide only a brief description of the algorithm and refer again to the articles [9] and [14] for more details. The algorithm performs series of iterations until satisfactory solution is found, meaning that some stopping rule, given lower and upper bound, is satisfied. We solve the following problem at stages $t = 2, \ldots, T$:

$$
\begin{aligned}
\min_{\boldsymbol{x}_t, u_t} \ & -\mathbf{1}^T \boldsymbol{x}_t + \lambda_{t+1} u_t + \theta_t \\
\text{s. t. } & \boldsymbol{p}_t^T \boldsymbol{x}_{t-1} - f_{tr}\mathbf{1}^T \boldsymbol{z}_t = \mathbf{1}^T \boldsymbol{x}_t \\
& \boldsymbol{z}_t \geq \boldsymbol{x}_t - \boldsymbol{x}_{t-1}, \ \boldsymbol{z}_t \geq \boldsymbol{x}_{t-1} - \boldsymbol{x}_t \\
& \theta_t \geq \hat{\mathcal{Q}}_t^j + \left(\mathbf{g}_t^j\right)^T (\boldsymbol{x}_t, \, u_t) \quad j = 1, \ldots, C \\
& \boldsymbol{x}_t \geq 0.
\end{aligned}
\tag{9}
$$

Here, θ_t in the objective function, coupled with cut constraints $\theta_t \geq \hat{\mathcal{Q}}_t^j + \left(\mathbf{g}_t^j\right)^T (\boldsymbol{x}_t, \, u_t)$, forms the outer linearization of the recourse function $\mathcal{Q}_{t+1}(\boldsymbol{x}_t, u_t)$. The structural and nonnegativity constraints simply repeat the same constraints from the model. In the final stage T cut constraints and θ_t term in objective function are omitted.

During a typical iteration of the SDDP algorithm, cuts have been accumulated at each stage. On a forward pass we sample a number of linear paths through the tree. As we solve a sequence of problems along these forward paths, the cuts that have been accumulated so far are used to form decisions at each stage. The solutions found form a policy which does not anticipate the future. The costs incurred along

all the sampled forward paths through the tree can be used to estimate the expected cost of the current policy, thus providing the upper bound.

In the backward pass of the algorithm, we add cuts to the collection defining the current approximation of the future cost functions. We do this by solving the descendant nodes of each node in the linear paths from the forward pass, except in the final stage T. To form a cut, we use the objective values and subgradients of the descendant nodes and employ the chain rule to calculate the subgradient of function $\mathcal{Q}_{t+1}(\boldsymbol{x}_t, u_t)$. The cuts collected at any node in stage t apply to all the nodes in that stage, therefore only one set of cuts in maintained for each stage. This complexity reduction is possible because of the interstage independence assumption. The optimal value of the first-stage problem provides the lower bound.

4   Empirical study

We used weekly data of the most important assets traded on the Prague Stock Exchange, November 2007 to March 2012. The week-to-week ratios were adjusted to include the stock dividends. The data summary can be found in the Table 1. We have fitted multidimensional correlated log-normal distribution to the adjusted price ratios, and the scenario tree was then constructed by sampling from this distribution, using the polar method for normal distribution sampling. The L'Ecuyer random generator was used to generate the required uniform random variables. Every computation including the sampling process was repeated 10 times, allowing to compute standard deviations of the solutions and objective values. There are many software products available for the purpose of stochastic optimization, for an overview see [6] or [16], but there is nothing available for the risk-averse version of the SDDP algorithm. Our computation was therefore implemented in own C++ software, using CPLEX to solve the required linear programs and Armadillo library for matrix computations. The confidence level was always set to 5%.

asset	mean	std. deviation
AAA	0.9980	0.0716
CETV	0.9929	0.0995
ČEZ	0.9994	0.0406
ERSTE GROUP BANK	0.9983	0.0795
KOMERČNÍ BANKA	1.0018	0.0543
ORCO	0.9899	0.0938
PEGAS NONWOVENS	0.9995	0.0398
PHILIP MORRIS ČR	1.0035	0.0368
TELEFÓNICA C.R.	1.0004	0.0266
UNIPETROL	0.9986	0.0506

Table 1: Data summary

We evaluated the model with two different settings of risk coefficients, $\lambda_t = \frac{1}{2}$ and $\lambda_t = \frac{t-1}{T}$. The first represents neutral and stable risk-aversion, while the latter means that we want to be sure about the final portfolio value by being more risk-averse as the final stage approaches. Both cases, with transaction costs of 0.3% and without transaction costs were considered. We have computed the optimal first-stage decisions for models with 2, 3 and 5 stages. In the following Table 2 we show the setup for the scenario trees used in our algorithm.

stages	descendants per node	total scenarios
2	$50,000$	$50,000$
3	$1,000$	$1,000,000$
5	$1,000$	10^{12}

Table 2: Testing problems setup

In all the testing cases, only three assets play a significant role in our portfolio: ČEZ, PHILIP MORRIS ČR and TELEFÓNICA C.R.. We will exclude the other assets from our results to ease the

orientation. We present the results without any transaction costs first, showing the optimal first-stage decisions for constant and growing risk-aversion settings in Tables 3 and 4. The difference between the optimal portfolios for 2, 3 or 5 stages is not significant in the constant risk coefficients setting. In the second case we can see a slight movement to the riskier asset in the first-stage decision, putting more weight to PHILIP MORRIS ČR. This is expected as the risk settings target mostly the expectation part of our recourse function in the first stage.

stages	ČEZ	PHILL	TELE
2	0.0663 (0.0087)	0.3169 (0.0081)	0.6168 (0.0092)
3	0.0510 (0.0459)	0.3112 (0.0537)	0.6273 (0.0707)
5	0.0450 (0.0307)	0.3340 (0.0268)	0.6043 (0.0571)

Table 3: Optimal decisions (std. deviations) with $f_{tr} = 0$ and $\lambda_t = \frac{1}{2}$

stages	ČEZ	PHILL	TELE
2	0.0663 (0.0087)	0.3169 (0.0081)	0.6168 (0.0092)
3	0.0597 (0.0645)	0.3429 (0.0650)	0.5792 (0.0920)
5	0.0392 (0.0415)	0.4325 (0.0678)	0.4975 (0.0652)

Table 4: Optimal decisions (std. deviations) with $f_{tr} = 0$ and $\lambda_t = \frac{t-1}{T}$

Next we show the optimal first-stage decisions with transaction costs of 0.3% in the Tables 5 and 6. It should be noted that in accordance with our model transaction costs have no effect in 2-stage models. We observe that the presence of the transaction costs reduces the differences found in the previous case with varying risk coefficients. We believe that this follows from the fact that varying risk coefficients require the investor to change the portfolio in every stage significantly. However, with the transaction costs in mind, this could be more expensive than the loss coming from holding slightly suboptimal, but stable portfolio. The impact of the transaction costs should be weaker in cases where stages cover longer time periods instead of just weeks.

stages	ČEZ	PHILL	TELE
2	0.0663 (0.0087)	0.3169 (0.0081)	0.6168 (0.0092)
3	0.0405 (0.0279)	0.2977 (0.0322)	0.6438 (0.0409)
5	0.0643 (0.0208)	0.3115 (0.0231)	0.6149 (0.0323)

Table 5: Optimal decisions (std. deviations) with $f_{tr} = 0.3\%$ and $\lambda_t = \frac{1}{2}$

stages	ČEZ	PHILL	TELE
2	0.0663 (0.0087)	0.3169 (0.0081)	0.6168 (0.0092)
3	0.0412 (0.0389)	0.3175 (0.0258)	0.6192 (0.0403)
5	0.0493 (0.0240)	0.3274 (0.0346)	0.6168 (0.0293)

Table 6: Optimal decisions (std. deviations) with $f_{tr} = 0.3\%$ and $\lambda_t = \frac{t-1}{T}$

The impact of adding more stages to the stage-independent model with constant risk-aversion settings tends to be minimal. This could, however, be different in the stage-dependent case. On the other hand, varying risk coefficients provide distinct solutions even with the independence assumption. In order to handle stage-dependent models, we would need to use different algorithm to solve the models and more complex estimation procedures.

5 Conclusion

We have developed a simple stock asset allocation model for the multistage setting and successfully employed the SDDP algorithm to solve it. We provide results based on assets from Prague Stock Exchange

for 2, 3 and 5-stage problems. Our results show that the nested model with constant risk-aversion coefficients provides similar results in both the 2-stage and multistage setting. On the other hand, it provides distinct behavior in the case when varying risk coefficients are used to represent more complicated investment strategies. We have also evaluated the effect of transaction costs in our model and pointed out that their presence could draw the dynamic model behavior closer to the static one.

The future experiments should include problems with more stages as well as different time periods represented by the stages, especially months and years. We would also like to focus on recent ways of measuring risk, see for example [1]. The stage-dependent case should also be covered, although it requires the use of different algorithm than SDDP.

Acknowledgements

References

[1] Branda M. and Kopa M.: DEA-risk efficiency and stochastic dominance efficiency of stock indices, *Czech Journal of Economics and Finance*, **62 (2)** (2012), 106–124.

[2] Dupačová, J., Consigli, G. and Wallace, S. W.: Scenarios for multistage stochastic programs, *Annals of Operations Research* **100** (2000), 25–53.

[3] Dupačová, J., Gröwe-Kuska, N., Römisch, W.: Scenario reduction in stochastic programming: An approach using probability metrics, *Mathematical Programming* **95** (2003), 493–511.

[4] Dupačová, J., Hurt, J. and Štěpán, J.: *Stochastic Modeling in Economics and Finance*, Kluwer Academic Publishers, 2002.

[5] Hoyland, K., Kaut, M. and Wallace, S.: A Heuristic for Moment-Matching Scenario Generation, *Computational Optimization and Applications* **24 (2-3)** (2003), 169–185.

[6] Kopa, M., ed.: *On Selected Software for Stochastic Programming*, MatfyzPress, Prague, 2008.

[7] Markowitz, H. M.: Portfolio Selection, *The Journal of Finance* **7 (1)** (1952), 77–91.

[8] Markowitz, H. M.: *Portfolio Selection: Efficient Diversification of Investments*, John Wiley & Sons, New York, 1959.

[9] Pereira, M. V. F. and Pinto, L. M. V. G.: Multi-stage stochastic optimization applied to energy planning, *Mathematical Programming* **52** (1991), 359–375.

[10] Pflug, G. Ch.: Scenario tree generation for multiperiod financial optimization by optimal discretization, *Mathematical Programming* **89** (2001), 251–271.

[11] Rockafellar, R.T. and Uryasev, S.: Optimization of Conditional Value-At-Risk, *The Journal of Risk* **2 (3)** (2000), 21–41.

[12] Ruszczynski, A. and Shapiro, A.: Conditional Risk Mappings, *Mathematics of Operations Research* **31 (3)** (2006), 544–561.

[13] Ruszczynski, A.: Risk-averse dynamic programming for Markov decision processes, *Mathematical Programming* **125 (2)** (2010), 235–261.

[14] Shapiro, A.: Analysis of stochastic dual dynamic programming method, *European Journal of Operational Research* **209** (2011), 63–72.

[15] Shapiro, A.: On a time consistency concept in risk averse multistage stochastic programming, *Operations Research Letters* **37** (2009), 143–147.

[16] Wallace, S. W., Ziemba, W. T., eds.: *Applications of Stochastic Programming*, SIAM - Mathematical Programming Society Series on Optimization, 2005.