Bootstrap application of the Bornhuetter-Ferguson method

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Abstract. The insurance companies use many different methods for calculation of the claim reserves, like the chain-ladder method, the Bornhuetter-Ferguson method, the Poisson method and many others. Majority of these models were originally created as deterministic models. Currently, these models are modified as stochastic models. When the stochastic model provides the relevant results, we have to dispose of information about the probability distribution of processed data. Due to real situation that the insurance companies do not publish the actual data, major complications arise both with coefficients estimates in the models and with determination of their properties (bias, standard deviation etc.). The bootstrap principle is used to estimate these coefficients in the paper; the described process does not require knowledge of the probability distribution of processed data, which can be considered as an advantage at solution of this problem.

Keywords: bootstrap principle, cumulative loses, expected ultimate loses, premium, Bornhuetter-Ferguson principle, insurance benefit, future loses.

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1 Introduction

The common situation in insurance practice is that the claim is not solved in the accident year but the insurance benefit is distributed in some following years. These events are divided into two basic groups that are connected with two types of claim reserves for past exposures, see Pacáková [6]:
a) IBNR (Incurred but not reported). The reserve for insurance benefit from the claims that have occurred but haven't been reported yet corresponds with it.
b) IBNS (Reported but not settled). This means not settled insured accident corresponding with the reserve for the insurance benefit from the claims that have been reported but have not been settled. The payment is expected in the future.

The insurance company has to create the adequate claim reserves for such situations. The table 1 describes the situation when we assume that the payment of the insurance benefit is distributed in n +1 years (development http://mme2012.opf.slu.cz/ .year).

accident	**development year *j***						
year *i*	**0**	**1**	**2**	**...**	***n*-2**	***n*-1**	***n***
0	$X_{0,0}$	$X_{0,1}$	$X_{0,2}$	**...**	$X_{0,n-2}$	$X_{0,n-1}$	$X_{0,n}$
1	$X_{1,0}$	$X_{1,1}$	$X_{1,2}$	**...**	$X_{1,n-2}$	$X_{1,n-1}$	$X_{1,n-1}$
2	$X_{2,0}$	$X_{2,1}$	$X_{2,2}$	**...**	$X_{2,n-2}$	$X_{2,n-1}$	$X_{2,n}$
...	**...**	**...**	**...**	**...**	…	…	…
***n* - 1**	$X_{n-1,0}$	$X_{n-1,1}$	$X_{n-1,2}$	**...**	$X_{n-1,n-2}$	$X_{n-1,n-1}$	$X_{n-1,n}$
n	$X_{n,0}$	$X_{n,2}$	$X_{n,2}$	**...**	$X_{n,n-2}$	$X_{n,n-1}$	$X_{n,n}$

Table 1 Insurance benefits

The first column of the table presents the year of the insurance event origin (accident year). Insurance benefits $X_{i,j}$ can be expressed as incremental values $Y_{i,j}$, or cumulative values $C_{i,j}$. The incremental insurance benefits $Y_{i,j}$ relating to the events that happened in the accident years $i = 0,1,2,\dots,n$, are settled in the years $j = 0,1,2,\dots,n$ (development year j). The cumulative insurance benefits $C_{i,j}$ present the sum of all insurance benefits in the de-

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velopment years 0 till j. The relation between incremental and cumulative insurance benefits has the following form $C_{i,j} = \sum_{k=0}^{j} Y_{i,k}$.

The insurance company knows the values lying over the main diagonal (the white cells). The main problem, the insurance company states ahead is to estimate the values lying in the cells below the main diagonal. These estimates are called in case of $Y_{i,j}$ the incremental claim reserves, in case of $C_{i,j}$ the cumulative claim reserves. The grey colour marks these cells in table 1.

Many methods deal with solution of this problem. Some of them are described for example in Bornhuetter, and Ferguson [1], Fecenko [2], Mack [4], Pacáková [5], Stehlík et al. [7], Wuttrich, and Merz [10]. The methods like loss-development method, chain ladder method, Cape-Cod method, additive method and others are classified as a special version of the Bornhuetter-Ferguson principle in the works made by Schmidt [8] and Schmidt, Zocher [9].

2 Bornhuetter – Ferguson method as general principle

Variables $Y_{i,j}$ and $C_{i,j}$ are considered to be random variables with an unknown distribution. General Bornhuetter – Ferguson principle assumes the existence of the parameters $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\gamma_1, \gamma_2, \dots, \gamma_n$ and it holds-true (Schmidt and Zocher [9])

$$E(C_{i,j}) = \alpha_i \gamma_j \qquad i, j \in \{1,2,...,n\} \tag{1}$$

where

$$\alpha_i = E(C_{i,n}) \quad \text{and} \quad \gamma_j = \frac{E(C_{i,j})}{E(C_{i,n})} \tag{2}$$

The unknown values $C_{i,j}$ are then predicted by a predictor

$$\hat{C}_{i,j} = \hat{\alpha}_i \, \hat{\gamma}_j \qquad i + j > n$$

where $\hat{\alpha}_i$ and $\hat{\gamma}_j$ are so called prior estimators of the parameters α_i and γ_j. These estimates can be obtained by different methods both from run-off triangle (upper triangle in the table 1), and from insurance market investigation. The difference $R_i = C_{i,n} - C_{i,n-i}$ is the accident year reserve and for its expected value can be written $E(R_i) = E(C_{i,n}) - E(C_{i,n-i})$. With regards to (2) this formula can be written in the form $E(R_i) = \alpha_i(1 - \gamma_{n-i})$. Then we use the estimator $\hat{R}_i$ for the estimate of the accident year reserve R_i

$$\hat{R}_i = \hat{\alpha}_i(1 - \hat{\gamma}_{n-i}) . \tag{3}$$

Schmidt [8] states that in the original Bornhuetter-Ferguson method for these coefficients holds true:

$$\hat{\alpha}_i = \pi_i \hat{\kappa}_i \ \text{ and } \ \hat{\gamma}_{n-i} = \hat{\gamma}_{n-i}^{CL} ,$$

where π_i is premium income of accident year i and $\hat{\kappa}_i$ is estimate of the expected loss ratio $\kappa_j = E\,[\,C_{i,n}/\pi_i\,]$ of the accident year i. $\hat{\gamma}_{n-i}^{CL}$ is the chain ladder estimator of parameter γ_{n-i} .For the cumulative losses is true $C_{i,k} = C_{i,n-i} + (C_{i,k} - C_{i,n-i})$. Similarly to the formula (3) when we express the expected value $E(C_{i,k} - C_{i,n-i})$ by the help of parameters α_i and γ_k we get the formula $E(C_{i,k} - C_{i,n-i}) = \alpha_i(\gamma_k - \gamma_{n-i})$. On the base of this formula Schmidt, and Zocher [9] define the general Bornhuetter-Ferguson predictor of future cumulative losses for i+$k > n$ as

$$\hat{C}_{i,k}^{BF} = C_{i,n-i} + \hat{\alpha}_i(\hat{\gamma}_k - \hat{\gamma}_{n-i}) \tag{4}$$

The general Bornhuetter - Ferguson predictor says that the future cumulative insurance benefits are compounded from the part of already paid insurance benefits and from the estimate of the insurance benefits in the development years n-i+1 till k.

Schmidt [8] shows that the different methods of the claim reserves estimates (loss-development method, chain-ladder method, Cape-Cod method and additive method) are basically the different variants of the general Bornhuetter - Ferguson principle (4). They differ only by the construction of the estimators $\hat{\alpha}_i$ and $\hat{\gamma}_j$.

3 Application of the bootstrap in special versions of Bornhuetter – Ferguson principle

Wuthrich, and Merz [10] state in their book the different methods of the claim reserves estimate and the different methods of calculation of their parameters.

The bootstrap approach to the parameters estimates is implied in this book. The more detailed description of this approach of the claim reserves estimate by Chain ladder method is in Linda, Kubanová, and Jindrová [3].

The succeeding methods, which are presented as the special versions of the general Bornhuetter-Ferguson principle, use the estimates of the development coefficients $\hat{\lambda}_j$, obtained by chain-ladder method. That is why the shortened description of this estimate by bootstrap method is stated.

The starting point is the AR model of the cumulative losses

$$C_{i,j+1} = \lambda_j \cdot C_{i,j} + \sigma_j \sqrt{C_{i,j}}\, \varepsilon_{i,j+1},$$

where $\varepsilon_{i,j+1}$ present a random residuals. The bootstrap replication $C^*_{i,j+1}$ for the upper triangle of the table 2 were gained by the help of resampling of the adjusted residuals

$$z_{i,j+1} = \hat{\varepsilon}_{i,j+1}\left(1 - C_{i,j}\left(\sum_{i=0}^{n-j-1} C_{i,j}\right)^{-1}\right)^{-\frac{1}{2}},$$

where $\hat{\varepsilon}_{i,j+1} = \hat{\sigma}_j^{-1}\left(\frac{C_{i,j+1}}{C_{i,j}} - \hat{\lambda}_j^{CL}\right)\sqrt{C_{i,j}}$ and $\hat{\lambda}_j^{CL} = \left(\sum_{i=0}^{n-j-1} C_{i,j+1}\right)\left(\sum_{i=0}^{n-j-1} C_{i,j}\right)^{-1}$. The bootstrap replications of the estimate of the development coefficient were calculated according to the formula

$$\hat{\lambda}_j^{CL*} = \left(\sum_{i=0}^{n-j-1} C^*_{i,j+1}\right)\left(\sum_{i=1}^{I-j} C^*_{i,j}\right)^{-1}, \text{ where } C^*_{i,j+1} = \hat{\lambda}_j C^*_{i,j} + \hat{\sigma}_j \sqrt{C^*_{i,j}}\, z^*_{i,j+1}.$$

When the chain ladder process in the general Bornhuetter – Ferguson model $\hat{C}_{i,k}^{BF} = C_{i,n-i} + \hat{\alpha}_i(\hat{\gamma}_k - \hat{\gamma}_{n-i})$ is used, then the bootstrap replication of the parameter $\hat{\gamma}_k$ is calculated according to the formula

$$\hat{\gamma}_k^{*CL} = \prod_{j=k+1}^{n} \frac{1}{\hat{\lambda}_k^{*CL}}. \tag{5}$$

Three different versions of the Bornhuetter – Ferguson method are presented in this paper.

3.1 Loss-development method

Loss development method estimator of future cumulative losses is given by formula

$$\hat{C}_{i,k}^{LD} = \hat{\gamma}_k \frac{C_{i,n-i}}{\hat{\gamma}_{n-i}}.$$

According to Schmidt [8] this formula can be expressed as a special version of the Bornhuetter – Ferguson principle

$$\hat{C}_{i,k}^{LD} = C_{i,n-i} + \frac{C_{i,n-i}}{\hat{\gamma}_{n-i}}(\hat{\gamma}_k - \hat{\gamma}_{n-i}) \qquad \text{where} \qquad \alpha_i = \frac{C_{i,n-i}}{\hat{\gamma}_{n-i}}.$$

Bootstrap replication of future cumulative losses are calculated according this formula as

$$\hat{C}_{i,k}^{*LD} = C_{i,n-i} + \frac{C_{i,n-i}}{\hat{\gamma}^*_{n-i}}(\hat{\gamma}^*_k - \hat{\gamma}^*_{n-i}) \tag{6}$$

where $\hat{\gamma}^*_i$ are given by formula (5).

3.2 Cape-Cod method

According to Schmidt [8] the Cape-Cod method predictions of the future cumulative losses are given by the formula

$$\hat{C}_{i,k}^{CC} = C_{i,n-i} + \pi_i \hat{\kappa}^{CC} (\hat{\gamma}_k - \hat{\gamma}_{n-i})$$

which is again the special version of the Bornhuetter-Ferguson principle, where $\hat{\alpha}_i = \pi_i \hat{\kappa}^{CC}$. The coefficient $\hat{\kappa}^{CC}$ is called the Cape-Cod loss ratio and it can be expressed by the formula

$$\hat{\kappa}^{CC} = \frac{\sum_{j=0}^{n} C_{j,n-j}}{\sum_{j=0}^{n} \hat{\gamma}_{n-j} \pi_j},$$

where π_i is the premium of the accident year.

When we use bootstrap for calculation of the future losses, we have to calculate at first the bootstrap replications of the Cape-Cod ratio $\hat{\kappa}^{*CC} = \frac{\sum_{j=0}^{n} C_{j,n-j}}{\sum_{j=0}^{n} \hat{\gamma}_{n-j}^{*} \pi_j}$, where $\hat{\gamma}_i^{*}$ are given by formula (5) again. Than the bootstrap predictor of the Cape Code method is

$$\hat{C}_{i,k}^{*CC} = C_{i,n-i} + \pi_i \hat{\kappa}^{*CC} (\hat{\gamma}_k^{*} - \hat{\gamma}_{n-i}^{*})$$

3.3 Additive method

The additive method predictors of the future cumulative losses have expression

$$\hat{C}_{i,k}^{AD} = C_{i,n-i} + \pi_i \sum_{j=n-i+1}^{k} \hat{\varsigma}_j^{AD}$$

where $\hat{\varsigma}_k^{AD} = \left(\sum_{j=0}^{n-k} X_{j,k} \right) \left(\sum_{j=0}^{n-k} \pi_j \right)^{-1}$. Schmidt [8] shows, that this model can be expressed in the Bornhuetter-Ferguson form as $\hat{C}_{i,k}^{AD} = C_{i,n-i} + \hat{\alpha}_i^{AD} (\hat{\gamma}_k^{AD} - \hat{\gamma}_{n-i}^{AD})$, where $\hat{\alpha}_i^{AD} = \pi_i \sum_{j=0}^{n} \hat{\varsigma}_j^{AD}$ and $\hat{\gamma}_k^{AD} = \left(\sum_{j=0}^{k} \hat{\varsigma}_j^{AD} \right) \left(\sum_{j=0}^{n} \hat{\varsigma}_j^{AD} \right)^{-1}$.

Bootstrap Additive method predictors are then

$$\hat{C}_{i,k}^{AD*} = C_{i,n-i} + \hat{\alpha}_i^{AD*} (\hat{\gamma}_k^{AD*} - \hat{\gamma}_{n-i}^{AD*})$$

where $\hat{\alpha}_i^{AD*} = \pi_i \sum_{j=0}^{n} \varsigma_j^{AD*}$, $\hat{\gamma}_k^{AD*} = \left(\sum_{j=0}^{k} \varsigma_j^{AD*} \right) \left(\sum_{j=0}^{n} \varsigma_j^{AD*} \right)^{-1}$ and $\hat{\varsigma}_k^{AD*} = \left(\sum_{j=0}^{n-k} X_{j,k}^{*} \right) \left(\sum_{j=0}^{n-k} \pi_j \right)^{-1}$.

$X_{j,k}^{*}$ are the bootstrap replications of the incremental losses, that were gained by the analogical process as the replications $C_{j,k}^{*}$.

4 Application of the Bornhuetter – Ferguson bootstrap principle

To demonstrate the Bornhuetter – Ferguson bootstrap principle in a concrete way, we used data published by Pacáková [5], that are shown in the following table 2. We can see in the tables 2 till 5 results of individual methods. The results of the common chain ladder method are presented in the table 2 (the lower dark triangle) and this table serves for comparison the results with results obtained by Bornhuetter – Fergusson bootstrap method. Result of bootstrap loss-development method (table 3), bootstrap Cape-Cod method (table 4) and bootstrap additive method (table 5), all after 1000 bootstrap simulation, are stated in this article.

Values of the predictors $\hat{\gamma}_k^{CL}$ and $\hat{\gamma}_k^{AD}$, that were calculated by classical chain-ladder and additive method, and the bootstrap replications $\hat{\gamma}_k^{*CL}$ of the parameter γ_k^{CL} are compared in the table 6. The bootstrap method enables to calculate bias of the parameter estimator of interest; it is stated in the last column of the table 6. The last table 7 summarizes the values of the predictor "the expected ultimate loss", they are calculated by classical chain-ladder and additive method (the first two columns) and by bootstrap method (the third and fourth column). We can see in all tables, that the results of individual methods are not fundamentally different; the bootstrap method enables us to avoid the assumptions that are necessary for application of the classical methods.

		development year *j*					
	i	**0**	**1**	**2**	**3**	**4**	**5**
accident year *i*	**0**	566	1 049	1 270	1 407	1 460	1 483
	1	501	993	1 186	1 345	1 409	1 431
	2	543	1 055	1 287	1 471	1 534	1 535
	3	652	1 323	1 633	1 842	1 921	1 922
	4	739	1 479	1 799	2 030	2 116	2 117
	5	752	1 478	1 479	1 669	1 740	1 741
		$\hat{\lambda}_0$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{\lambda}_4$	
		1.966	**1.216**	**1.128**	**1.043**	**1.016**	

Table 2 Cumulative values of claim reserves

		development year *j*						$\hat{\alpha}_i^{LD}$
	i	**0**	**1**	**2**	**3**	**4**	**5**	
accident year *i*	**0**	566	1 049	1 270	1 407	1 460	1 483	1483
	1	501	993	1 186	1 345	1 409	1 431	1431
	2	543	1 055	1 287	1 471	1 534	1 534	1557
	3	652	1 323	1 633	1 837	1 921	1 915	1944
	4	739	1 479	1 794	2 102	2 116	2 103	2135
	5	752	1 469	1 470	1 722	1 740	1 723	2120

Table 3 Bootstrap loss-development method

		development year *j*						$\hat{\alpha}_i^{CC}$	π_i
	i	**0**	**1**	**2**	**3**	**4**	**5**		
accident year *i*	**0**	566	1 049	1 270	1 407	1 460	1 483	1453	1 700
	1	501	993	1 186	1 345	1 409	1 431	1436	1 680
	2	543	1 055	1 287	1 471	1 532	1 556	1538	1 800
	3	652	1 323	1 633	1 830	1 901	1 934	1880	2 200
	4	739	1 479	1 782	1 996	2 078	2 109	2051	2 400
	5	752	1 330	1 582	1 761	1 826	1 855	1709	2 000

Table 4 Bootstrap Cape-cod method

		development year *j*						$\hat{\alpha}_i^{LD}$	π_i
	i	**0**	**1**	**2**	**3**	**4**	**5**		
accident year *i*	**0**	566	1 049	1 270	1 407	1 460	1 483	1505	1 700
	1	501	993	1 186	1 345	1 409	1 432	1487	1 680
	2	543	1 055	1 287	1 471	1 533	1 558	1593	1 800
	3	652	1 323	1 633	1 837	1 913	1 943	1948	2 200
	4	739	1 479	1 790	2012	2 095	2 128	2125	2 400
	5	752	1 345	1 604	1 789	1 858	1 885	1771	2 000

Table 5 Bootstrap additive method

$\hat{\gamma}_k^{CL}$	$\hat{\gamma}_k^{AD}$	$\hat{\gamma}_k^{*CL}$	**bias** $\hat{\gamma}_k^{CL}$
0.350	0.360	0.355	0.005
0.688	0.695	0.693	0.004
0.837	0.841	0.840	0.003
0.944	0.946	0.945	0.001
0.984	0.985	0.984	0.000
1.000	1.000	1.000	0.000

Table 6 The calculated $\hat{\gamma}_k^{CL}$, $\hat{\gamma}_k^{AD}$ and bootstrap $\hat{\gamma}_k^{*CL}$ values of the predictors

$\hat{\alpha}_i^{LD}$	$\hat{\alpha}_i^{CC}$	$\hat{\alpha}_i^{*LD}$	$\hat{\alpha}_i^{*CC}$
1483	1456	1483	1453
1431	1439	1431	1436
1558	1542	1557	1538
1951	1884	1944	1880
2149	2055	2135	2051
2148	1713	2121	1709

Table 7 The original (left) and bootstrap (right) values of the expected ultimate loss

5 Conclusion

When the stochastic version of the Bornhuetter - Fergusson principle is used for calculation of the claims, the problems caused by lack of knowledge of the probability distribution of the data arise. The origin of mentioned problems is usually the small number of the data. In such cases the bootstrap method can be applied, principle of which is relatively simple without presumption of knowledge of probability distributions. The authors presented three versions of application of Bornhuetter - Fergusson principle for claims reserves calculation with the concrete data in this article. The results shown in the tables 3 - 7 don´t differ significantly each other; we can conclude that the application of the bootstrap for prediction of the claim reserves provides reasonable results and thus it can be used in practice. Of course, as with all the methods, it is necessary to confront the results with other methods.

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