The Ordinal Consensus Ranking Problem with Uncertain Rankings

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Abstract: In the ordinal consensus ranking problem (OCRP) a set of k decision makers rank a set of n alternatives with regard to one overall criterion (or a set of criteria) from the 1st place to the n^{th} place. The goal is to find a consensus ranking expressing an opinion of a group. The aim of this article is to present a model for OCRP solution with uncertain rankings. This approach is more suitable than classic approach with certain rankings, as the latter case doesn't allow for imprecise information or uncertainty often involved in real decision-making processes. In this paper uncertain ranking g_{ij} is defined as a decision maker's confidence that an alternative i is ranked at the j^{th} position, where $g_{ij} \in [0,1]$ and $\sum_j g_{ij} = 1$ for all i. In the model for OCRP solution with uncertain rankings generalized means operator is used for ranking aggregation and the final consensus ranking is obtained by the use of a binary dominance relation. The model can be extended to multiple criteria or different weights of decision makers, and it can handle the cases with certain rankings as well. Also, the model's setting enables to evaluate decision makers' preferences in terms of inconsistence and indecisiveness.

Keywords: group decision making, ordinal consensus ranking problem, preference ranking, uncertain ranking.

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1 Introduction

The ordinal consensus ranking problem (OCRP) represents a special case of (multi-criteria) group decision making which history dates back to the works by Borda [2] and Condorcet [3] from the late 18th century. In OCRP a set of decision makers (experts) rank a set of alternatives with regard to a given set of criteria or one overall criterion. The goal is to find a consensus ranking expressing an opinion of a group. In general, there are two different classes of (classic) methods for OCRP solution, ad-hoc methods and distance based, which are briefly discussed in the next Section. The 'state of the art' of the ordinal consensus rankings problem can be found in [4]. Recently, some new methods of solution were proposed (see [11, 13]) and research has focused on examination of conditions, under which the same result is obtained by different methods ([6, 7, 8, 12, 13]).

The aim of this article is to present a model for a solution to ORCP when alternatives' rankings are uncertain. This approach may be more suitable when compared to the classic approach with certain rankings, as the latter case doesn't allow for imprecise information or uncertainty often involved in real decision making processes. In OCRP it is assumed that each decision maker provides rank order (ranking) of all alternatives. However, in many cases a decision maker is not able to do so due to lack of knowledge, time pressure, imprecise information, etc. In such situations a decision maker can express his preferences in a form of ranking with some degree of uncertainty. If a decision maker is certain about his ranking, he assigns each alternative its position with full confidence (with the probability equal to 1). But if he is uncertain, he can give a degree of confidence that an alternative is placed at the n^{th} position by the number from [0,1].

The proposed model with uncertain rankings can be extended for multiple criteria and different weights of decision makers, and it incorporates cases with crisp rankings as well, thus providing a generalization to classic methods for OCRP solution. Moreover, model's setting enables to evaluate 'quality' of decision makers' preferences in terms of indecisiveness and inconsistence.

The paper is organized as follows: the classic approach to OCRP is discussed in Section 2; the proposed model with uncertain rankings is described in Section 3 followed by an illustrative example in Section 4. Model's extensions are presented in Sections 5 and 6, and Conclusions close the article.

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2 Methods for OCRP solution with certain rankings

In OCRP with certain rankings each decision maker (DM) ranks alternatives from the best to the worst. Each ranking constitutes total order on a set of alternatives, as each alternative is assigned precisely one position. The final group ranking (consensus) is achieved by some of these classic single criterion methods:

- **Ad-hoc methods**

-*Borda-Kendall's method of marks* (see [2] and [9]): each alternative is given a number of points corresponding to its rank. The best alternative is the alternative with the lowest total count (mark) or the lowest average.
-*Condorcet's simple majority rule* (see [3]): the best alternative is an alternative preferred over all other alternatives in pair-wise comparisons.
-*Maximize agreement heuristic* (MAH) by Beck and Lin [1]: all alternatives are pair-wise compared by all DMs and then ranked in the descending order according to their total number of preferences (*P*) or non-preferences (*N*).

- **Distance based methods**

In distance based methods rankings of DMs are converted into a vector, object-to-object or object-to-rank matrix (alternatives are displayed in rows and their position in columns) representation subsequently. Then, by the use of a suitable distance function on vector or matrix spaces, the consensus is searched through the space of all possible rankings (permutations of the order *n*), where the consensus is defined as the ranking with the minimal sum of distances to rankings of all DMs. Usually, the following l_l metric is used as a distance function:

$$d(A,B) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| a_{ij} - b_{ij} \right|,$$

where A (a_{ij}) and B (b_{ij}) are square matrices of the order n. Distance based methods include e.g. *Consensus ranking model* (*CRM*) by Cook and Kress [5] and *Distance-based ideal-seeking consensus ranking model* (*DCM*) by Tavana et al. [13].

However, all aforementioned methods share several limitations. They cannot handle ties or non-preferences between alternatives; they don't allow expressing a degree of preference among alternatives; they don't enable to express decision makers' importance (weight) and finally they assume precise information in the form of certain ranking of alternatives is provided by DMs. Above mentioned disadvantages can be put aside by the use of the proposed model with uncertain rankings.

3 OCRP with uncertain rankings

3.1 Uncertain rankings

In the context of this paper certain rankings (briefly c-rankings) have to be distinguished from uncertain rankings (u-rankings). C-ranking is represented by a binary preference matrix with rows corresponding to alternatives and columns to positions (for an example see Figure 1). These matrices are bistochastic, as there is precisely one 1 on each row and column. By analogy, u-ranking can be represented by a row stochastic matrix with elements in the interval $[0,1]$, see Figure 1. Formally, u-rankings are introduced by the following Definition 1.

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 0.3 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \end{pmatrix}$$

Figure 1 Matrix representation of c-rankings (K) and u-rankings (L) of four alternatives A, B, C, and D. Matrix K gives following crisp ranking: (A, B, D, C)

Definition 1. *Let $g_{ij}^{k} \in [0,1]$ be confidence of the k^{th} decision maker that an alternative i occupies the j^{th} position, where i, j ∈ {1,2,...,n} and k ∈ {1,2,...,K}. Then $g_{ij}^{k} \in [0,1]$ is called uncertain ranking (u-ranking), if:*

$$\sum_{j=1}^{n} g_{ij}^{k} = 1 \text{ for all } i \text{ and } k \qquad (1)$$

According to this definition u-rankings are normalized. Ordering of alternatives is uncertain (fuzzy) in the sense that each alternative can occupy each position, but generally with the different degree of confidence (see Table 1). Hence, uncertain rankings can be regarded as fuzzy measures on the set of positions. From a probability point of view, u-ranking of a given alternative can be interpreted as a probability mass function that assigns a value $p(x_i)$ to each position x_i, $i \in \{1, 2, ..., N\}$, such that:

$$p\left(x_i\right) \geq 0 \text{ and } \sum_{i=1}^{N} p\left(x_i\right) = 1 \tag{2}$$

3.2 Aggregation of uncertain rankings

DMs' u-rankings g_{ij}^{k} have to be aggregated by aggregation functions or operators. In this paper generalized means [16] $h : [0,1]^n \to [0,1]$ are used in the following form:

$$h_{ij}(g_{ij}^{(k)}, \alpha) = \left[\frac{\sum_{k=1}^{K} g_{ij}^{(k)\alpha}}{K} \right]^{1/\alpha}, \tag{3}$$

where $\alpha \in R - \{0\}$. For $\alpha = 1$ we obtain the arithmetic mean, for $\alpha \to 0$ the geometric mean and for $\alpha = -1$ the harmonic mean [10]. In the model the averaging operator (3) with $\alpha = 1$ and $\alpha \to 0$ is used. The average (group) u-ranking of an alternative *i* at the position *j* is denoted as h_{ij}.

3.3 Ordering of alternatives

As the aim of OCRP is to establish the final group consensus ranking, alternatives must be compared and ordered finally. For a comparison of alternatives the following binary dominance relation is introduced:

Definition 2. *Let h_{ij} be the group u-ranking of the alternative i at the position j. Then, the cumulative group u-ranking H_{ij} of alternative i from the 1st to the jth position is given as:*

$$H_{ij} = \sum_{k=1}^{j} h_{ik} \tag{4}$$

Definition 3. *An alternative r dominates an alternative s $\left(r \succ s\right)$ if cumulative group u-rankings H_{rj} of an alternative r are at least equal to cumulative group u-rankings H_{sj} of an alternative s for every position j, and there is a position p such that cumulative group u-ranking H_{rp} is higher than H_{sp}:*

$$r \succ s \Leftrightarrow H_{rp} \geq H_{sp}, \forall n \in \{1, 2, ..., N\} \wedge H_{rp} > H_{sp}, \qquad 1 \leq p \leq N \tag{5}$$

The dominance relation (5) provides a partial quasi-order on the set of alternatives, as some alternatives might not be comparable and thus the final consensus ranking (and the best alternative) might not be unique.

3.4 The model

The proposed model for the solution of the ordinal consensus ranking problem with u-rankings is composed of three parts: u-rankings of individual decision makers, the aggregation (averaging) operator and the dominance relation.

Decision makers' uncertain rankings of each alternative, preferably in a matrix format, represent the model's input. The output of the model is the best alternative (or alternatives). The model proceeds in five steps:

1. Each DM gives u-ranking g_{ij} for each alternative according to his knowledge and confidence.
2. DMs' u-rankings are aggregated for each alternative and each position by the averaging operator (3). When other than arithmetic mean is used for aggregation, average group u-rankings have to be normalized subsequently.
3. For each alternative *i* cumulative group u-rankings H_{ij} is evaluated by (4). Because of normalization H_{in} = 1 for each alternative *i*.
4. All alternatives are pair-wise compared with the use of the dominance relation (5).
5. Alternatives are ranked according to their dominance.

It is possible to integrate an additional step between steps 1 and 2 evaluating decision makers' rankings in terms of indecisiveness and inconsistence (see Section 6 for details).

As certain rankings constitute only a special case (a subset) of uncertain rankings, they can be handled by the model as well. The next section illustrates how the model works.

4 Illustrative example

Four decision makers (DM_1 to DM_4) rank four alternatives A, B, C and D from the best to the worst. U-rankings of DMs are presented in Table 1. Rankings for each alternative are averaged with respect to DMs via relation (6) with n = 4, K = 4 and α = 1 and they are shown in Table 2. Cumulative group u-rankings of all alternatives are presented in Table 3.

As for alternatives' comparison, from the dominance relation (7) we get:

$$A \succ B,\ A \succ C,\ A \succ D,\ B \succ C,\ B \succ D$$

Alternatives C and D are non-comparable. Therefore, we obtain two final rank orders: (A, B, C, D) and (A, B, D, C). In both cases the best alternative is A.

If the geometric mean ($\alpha \to 0$) is used for rankings aggregation instead of the arithmetic mean, results wouldn't change (see Table 4 and Table 5). However, the geometric mean is not appropriate operator for aggregation of rankings consisting of many 0.

DM_1	1^{st}	2^{nd}	3^{rd}	4^{th}	DM_2	1^{st}	2^{nd}	3^{rd}	4^{th}
A	0.4	0.2	0.1	0.3	A	0.5	0.3	0.1	0.1
B	0.3	0.3	0.2	0.2	B	0.3	0.3	0.2	0.2
C	0.1	0.2	0.4	0.3	C	0.2	0.3	0.4	0.1
D	0.1	0.4	0.4	0.1	D	0.1	0.3	0.5	0.1
DM_3	1^{st}	2^{nd}	3^{rd}	4^{th}	DM_4	1^{st}	2^{nd}	3^{rd}	4^{th}
A	0.6	0.2	0.1	0.1	A	0.4	0.4	0.2	0
B	0.3	0.4	0.2	0.1	B	0.3	0.3	0.3	0.1
C	0.2	0.2	0.3	0.3	C	0.1	0.2	0.4	0.3
D	0.1	0.2	0.3	0.4	D	0.1	0.3	0.4	0.2

Table 1 U-rankings of decision makers DM_1 - DM_4 for alternatives A, B, C, D

Alternative	1^{st}	2^{nd}	3^{rd}	4^{th}
A	0.475	0.275	0.125	0.125
B	0.3	0.325	0.225	0.15
C	0.5	0.225	0.375	0.25
D	0.1	0.3	0.4	0.2

Table 2 Group u-rankings of alternatives A, B, C, D for the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} place

Alternative	1^{st}	2^{nd}	3^{rd}	4^{th}
A	0.475	0.75	0.875	1
B	0.3	0.625	0.85	1
C	0.15	0.375	0.75	1
D	0.1	0.4	0.8	1

Table 3 Cumulative group u- rankings of alternatives A, B, C, D for all places

Alternative	1^{st}	2^{nd}	3^{rd}	4^{th}
A	0.564	0.3	0.136	0
B	0.305	0.327	0.225	0.144
C	0.147	0.23	0.387	0.237
D	0.105	0.306	0.413	0.176

Table 4 The geometric mean of uncertain rankings of alternatives A, B, C and D

Alternative	1^{st}	2^{nd}	3^{rd}	4^{th}
A	0.564	0.864	1	1
B	0.305	0.632	0.857	1
C	0.147	0.377	0.764	1
D	0.105	0.411	0.824	1

Table 5 Cumulative group u-rankings of alternatives A, B, C and D

5 Extensions

The model's setting presented in Section 3 enables straightforward extensions in terms of decision makers' weights and multiple criteria:

- To each decision maker weights w_i can be assigned according to his/her importance or knowledge. For the aggregation of preferences, e.g. the weighted arithmetic mean can be used:

$$h_{ij}\left(g_{ij}^{(k)}, w_k\right) = \left(\sum_{k=1}^{K} g_{ij}^{(k)} \cdot w_k\right) / \sum_{k=1}^{K} w_k \tag{6}$$

- Alternatives can be ranked by more than one criterion, and in this case criteria themselves can be ranked in order of importance in the same way as alternatives. The overall u-ranking of each alternative is obtained by the aggregation over criteria of each DM and then over all decision makers (or vice versa) with the use of (6). Again, u-rankings should be normalized in the process.

6 The evaluation of decision makers' preferences

The model's framework allows evaluating experts' decisions in terms of *indecisiveness* and *inconsistence*. An expert is absolutely decisive, when he assigns each alternative value 1 for a given position and value 0 to all other positions, and indecisive otherwise. To evaluate indecisiveness, Shannon's entropy as a measure of uncertainty can be used [10]:

$$H\left(p(x)\right) = -\sum_{i=1}^{N} p\left(x_i\right)\log_2 p\left(x_i\right), \tag{7}$$

where $p(x_i)$ are probabilities assigned to values x_i, $i \in \{1, 2, ...N\}$; and $H\left(p\left(x_i\right)\right) = 0$ for $p\left(x_i\right) = 0$.

A decision maker is absolutely indecisive, if he provides u-rankings with the uniform distribution $p\left(x_i\right) = \frac{1}{N}$, $i \in \{1, 2, ...N\}$ for a given alternative (see an example on the left-hand side of Table 6). In this case, the entropy (7) is equal to the Hartley's information I (N) (Hartley's measure of *nonspecifity*) $I\left(N\right) = \log_2 N$. Because each decision maker provides u-rankings of N alternatives, DM's maximum indecisiveness IND_{max} is given as:

$$IND_{\max} = N \log_2 N \tag{8}$$

The overall DM's indecisiveness IND is given as:

$$IND = -\sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij} \cdot \log_2\left(g_{ij}\right) \tag{9}$$

A DM is absolutely consistent in his judgment, if his sum of u-rankings for each position over all alternatives is 1, and inconsistent otherwise. Therefore, inconsistence IND in the model's setting is given as:

$$INC = \sum_{j=1}^{N}\left|\sum_{i=1}^{N} g_{ij} - 1\right| \tag{10}$$

Maximum inconsistence $INC_{\max}$ is achieved when a DM assigns value 1 to the same position for all alternatives (see an example on the right-hand side of Table 6). Then from (10) we obtain:

$$INC_{\max} = 2\left(N - 1\right) \tag{11}$$

Relations (8-9) and (10-11) allow expressing the relative indecisiveness IND_{r} and reative inconsistence INC_{r}:

$$IND_r = \frac{IND}{IND_{\max}} \tag{12}$$

$$INC_r = \frac{INC}{INC_{\max}} \tag{13}$$

Extreme cases of experts' decisions are illustrated in Table 6. DM_1 is absolutely indecisive (IND = 2), but he is absolutely consistent (INC = 0), while DM_2 is absolutely decisive (IND = 0), but he is absolutely inconsistent (INC = 6). Unlike many other models, which pay little or no attention to the quality of experts' decisions, in the presented model experts' decisions can be easily and clearly scrutinized in terms of indecisiveness and inconsistence, and highly indecisive and/or highly inconsistent experts (such as DMs shown in Table 6) might be given lower weights or even may be excluded from a decision making process.

DM_1	1st	2nd	3rd	4th	DM_2	1st	2nd	3rd	4th
A	0.25	0.25	0.25	0.25	A	0	1	0	0
B	0.25	0.25	0.25	0.25	B	0	1	0	0
C	0.25	0.25	0.25	0.25	C	0	1	0	0
D	0.25	0.25	0.25	0.25	D	0	1	0	0

Table 6 DM_1 assigns each alternative and each position the same value 0.25, hence he is absolutely indecisive. DM_2 ranks all alternatives in 2nd position, and hence he is absolutely inconsistent

7 Conclusions

The aim of the article was to present a simple model for ordinal consensus ranking problem with uncertain rankings, and to illustrate the use of the model by examples. The model is more realistic for a solution of real-world problems involving uncertainty and imprecise information. Other advantages of the model include computational simplicity and extensions to multiple-criteria or different weights of decision makers. Moreover, in the model's setting experts' judgments can be evaluated in terms of indecisiveness and inconsistence. As certain rankings constitute the subset of uncertain rankings, the model provides generalization to classic methods for the ordinal consensus ranking problem solution.

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