Modeling dependence and feedback in ANP with fuzzy cognitive maps

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Abstract: The objective of multi-criteria decision making is to select the best alternative from a set of feasible alternatives with regard to a given set of criteria. To include dependence and feedback among criteria analytic network process (ANP) was proposed by T. L. Saaty. However, ANP have two disadvantages: firstly, it is difficult to provide correct network structure among criteria even for experts, and different structures lead to different results. Secondly, to form a supermatrix all criteria have to be pair-wise compared with regard to all other criteria, which is difficult and also unnatural. To circumvent these problems criteria network structure in ANP can be modeled with fuzzy cognitive maps. The aim of this article is to propose the hybrid eigenvalue-fuzzy cognitive map method (HEFCM) for the derivation of criteria weights. In the first step of HEFCM initial (local) weights of criteria are determined by Saaty's eigenvalue method. In the second step contribution to criteria weights from dependence and feedback is established by fuzzy cognitive map approach. Final (global) weights of all criteria are obtained by an aggregation of both weights. The proposed method is illustrated by an example.

Keywords: Analytic hierarchy process (AHP), analytic network process (ANP), criteria weights, decision making, fuzzy cognitive maps

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1 Introduction

The objective of multi-criteria decision making (MCDM) is to select the best alternative or object from a set of feasible alternatives or objects with regard to a given set of criteria. However, in real-world MCDM problems criteria (and other elements) are not independent as they influence each other. To include dependence and feedback into consideration analytic network process by T. L. Saaty was proposed (see [4, 5]).

However, ANP has two disadvantages [6]: firstly, it is difficult to provide a correct network structure even for experts, and different structures lead to different results. Secondly, to form a supermatrix all criteria have to be pair-wise compared with regard to all other criteria, which is also difficult and somewhat unnatural, as we ask themselves questions of the type: "How much is a criterion A more important than a criterion B with regard to a criterion C?" To circumvent these two problems criterias' network structure in ANP can be modeled with fuzzy cognitive maps.

Fuzzy cognitive maps are graphical tools introduced by B. Kosko [2] enabling to express dependence and feedback among concepts with different intensity given by a real number from [–1,1] or [0,1] interval. Fuzzy cognitive maps were found useful in many decision making areas such as politics, management, environmental protection or medicine.

The aim of this article is to propose and illustrate the use of the hybrid eigenvalue-fuzzy cognitive map method (HEFCM) for the derivation of global weights of criteria under dependencies and feedbacks based on Saaty's eigenvalue method and fuzzy cognitive maps, which can circumvent two disadvantages of ANP mentioned before. This new approach is compared to ANP in an illustrative example.

The paper is organized as follows: in section 2 AHP/ANP is briefly described, in section 3 fuzzy cognitive maps are introduced, section 4 provides description of our HEFCM method and in section 5 a numerical example is provided. Conclusions close the article.

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2 Analytic hierarchy process (AHP) and analytic network process (ANP)

In AHP hierarchical structure of elements such as goal, criteria (sub-criteria) or alternatives is considered, where elements from higher levels of hierarchy influence elements from (immediately) lower levels, but not vice versa, and elements on the same level are considered independent. In this paper we limit ourselves to the classic 3-level hierarchy (goal, criteria and alternatives).

The basis of AHP/ANP is a pair-wise comparison of elements: the relative importance of elements from a given level of hierarchy with regard to an element on immediately higher level is expressed by a number on Saaty's fundamental scale (see Table 1). To derive weights of criteria, the pair-wise comparison matrix S (s_{ij}) is constructed, where $s_{ij} = v_i / v_j$ is the ratio of importance of an element i compared to an element j, $s_{ij} \in \{1,2,3,4,5,6,7,8,9\}$. Because $s_{ij} = 1/s_{ji}$ for all i and j, the matrix S is reciprocal (see Figure 1).

Weights of criteria (w) are determined by a principal eigenvector belonging to the largest (positive) eigenvalue of S, hence the vector of weights w satisfies the equation $Sw = \lambda_{\max} w$. The existence of the largest eigenvalue is guaranteed by Perron-Frobenius theorem. Let f_j be criteria, let $v_i(f_j)$ be a weight of an alternative i with regard to a criterion j, and let w_i be weights of criteria with regard to the goal, then the weight of an alternative i with regard to the goal is given as $\sum_j v_i(f_j) \cdot w_j$. For the optimal alternative this values attains its maximum.

However, in many real-world situations criteria or alternatives might be interacting and influencing one another. In analytic network process elements are divided into clusters and these clusters form network structures with dependence and feedback. In the first step of ANP elements (criteria) from one cluster are pair-wise compared with regard to elements (criteria) from other clusters, and a supermatrix W is formed. The supermatrix is a block matrix consisting of matrices W_{ij}, where columns of W_{ij} express the relative importance (priority) of elements from a cluster i to elements from a cluster j (columns of W_{ij} are obtained from pair-wise comparisons in the form of eigenvectors). If $W_{ij} = 0$ then a cluster i has no influence on a cluster j.

Intensity of importance	Definition
1	Equal importance
2	Weak
3	Moderate Importance
4	Moderate plus
5	Strong Importance
6	Strong plus
7	Very strong Importance
8	Very, very strong
9	Extreme importance

Table 1 Saaty's fundamental scale. Source: [3]

An element $w_{mn} \in W$ determines direct influence of an element m on an element n. Indirect influences can be determined by raising the supermatrix to the powers, and the final influence is obtained by the limiting process $\lim_{n \to \infty} W^n = W^\infty$ (the weights are in columns). For the convergence (or at least cyclicity) of a supermatrix the stochasticity of W is required.

A structure of the limiting supermatrix W^∞ depends on a network structure (for details see [3] or [4]). For instance, if a goal influences criteria, and criteria influence alternatives as well as criteria themselves, then the supermatrix W has the following form:

$$W = \begin{array}{c} \\ G \\ C \\ A \end{array} \begin{array}{c} \begin{array}{ccc} G & C & A \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ W_{21} & W_{22} & 0 \\ 0 & W_{32} & I \end{bmatrix} \end{array},$$

where W_{22} expresses dependencies among criteria. The priority vector (vector of weights) Q of alternatives is obtained from the limiting supermatrix W^∞ in the block W_{31}^∞, which has the form [3]:

$$Q = W_{32}(I - W_{22})^{-1} W_{21}$$

Given priorities Q all alternatives can be sorted from the best to the worst.

3 Fuzzy cognitive maps

Cognitive maps (CP) were introduced in 1976 by Axelrod [1] in the context of social and politic decision making. CP consist of concepts (nodes) describing system's behavior connected by edges (arcs) representing their causal relationship. Edges can be assigned three values: 0 (no relationship between two concepts), –1 or '–' (a negative relationship) and 1 or '+' (a positive relationship). In the graph theory nomenclature CP belong among directed graphs.

Fuzzy cognitive maps (FCM), also called fuzzy decision maps, were proposed by B. Kosko in [2], and they can be considered a combination of the fuzzy set theory and neural networks. As relationships among concepts can be 'fuzzy', FCM allow expressing a degree of an influence of one concept on another. To express the degree of an influence of a concept i on a concept j, the mapping $e_{ij} \to [0,1]$ or $e_{ij} \to [-1,1]$ is used, where the higher absolute values of e_{ij} denote the higher influence (the stronger causal relationship). In this paper we define FCM as follows:

Definition 1. *FCM is a tuple (C, E), where C is a set of concepts with cardinality n and E is a square adjacency-matrix of the order n with elements* $e_{ij} \in [0,1]$*, where* e_{ij} *expresses the strength of an influence of a concept i on a concept j.*

Usually, diagonal elements $e_{ii} = 0$ as it is assumed that concepts cannot influence themselves, the rule which we observe too.

In our approach within AHP/ANP framework we use criteria as concepts and their dependence and feedback is modeled by edges representing their relationships. In the following section we provide description of our method for the derivation of criterias' weights.

4 The hybrid eigenvalue-fuzzy cognitive map method for the derivation of global weights of criteria

To derive weights of criteria with interdependence and feedback we propose HEFCM (hybrid eigenvalue-fuzzy cognitive map) method, which proceeds in the following steps:

1. All criteria are pair-wise compared and a reciprocal matrix S is formed. Weights (also called local weights) w of criteria (without dependence and feedback) are established by the standard eigenvalue method. Also, all alternatives are pair-wise compared with regard to all criteria by Saaty's method and their weights are established (these weights in a matrix format correspond to the matrix W_{32} of the supermatrix W in ANP).
2. Fuzzy cognitive map of dependencies and feedbacks among criteria is established and turned into the adjacency (connection) matrix E with elements $e_{ij} \in [0,1]$.
3. Matrix E is converted into normalized matrix E^* with all column sums equal to 1, which is called column stochastic. Column stochasticity of the matrix E^* is necessary for the convergence in the Step 4.
4. The influence of an element i on the element j is given by multiplication of E^*, that is:

$$\lim_{n \to \infty} (E^*)^n = M \tag{1}$$

The convergence of a limit (1) is ensured by column stochasticity of a matrix E^*. If there is no single limit, but a limit cycle consisting of k matrices E_k^*, then the limiting supermatrix M is given by the Césaro summation:

$$\lim_{n \to \infty} \frac{1}{k} \sum_{i=1}^{k} (E_k^*)^n = M \tag{1'}$$

5. The weights are aggregated by the following formula:

$$v = f(w, m) \tag{2}$$

where m is a column vector of M, and $f(w,m)$ is an aggregation operator. Here, we use the arithmetic mean: $v_i = \frac{1}{2}\sum_{i=1}^{n}(w_i + m_i)$. This approach is equivalent to the aggregation formula $v = w + E^* \cdot w$ proposed in [6].

6. The priority vector (global weights) Q of all alternatives with regard to the goal is obtained by the formula:

$$Q = W_{32} \cdot v^T \quad (3)$$

One important advantage of the proposed method is that it does not require threshold functions (logistic, hyperbolic-tangent, sigmoid, etc., see [2] or [6]). In the FCM approach the matrix E is multiplicated by state vectors C_n to (often) generate vector elements larger than 1 or lower than 0, which must be 'cut off' to 1 or 0, respectively, by threshold functions. As a consequence, the result is strongly dependent on a particular threshold function used, see e.g. [6]. In our approach the problem with threshold functions is avoided by the column stochasticity of the matrix E^*. It is worth noting that the very same condition is applied in ANP for the supermatrix W, as it is crucial for the existence of the limiting supermatrix W^∞.

5 The numerical example

In this section we provide the illustration of HEFCM method and its comparison with ANP. In our example we are going to select the best car from two alternatives A and B, when the following interacting criteria for the selection are considered: price (P), safety (S), design (D), equipment (E), and fuel consumption (F). Dependencies among criteria are shown in Figure 4 b).

5.1 HEFCM solution

Step 1: We compare all criteria pair-wise to obtain weights of criteria without dependence and feedback. Results of the comparison are presented in Figure 1. From Figure 1 we derive (normalized) weights of criteria (in the order: P, S, D, E, F) by the eigenvalue method ($\lambda_{\max} = 5.1182$, I.C. = 0.03): w = (0.417, 0.282, 0.153, 0.098, 0.050). Then we compare pair-wise all alternatives with regard to all criteria, see Figure 2 a), and obtain weights of criteria (see Figure 2 b)).

Step 2: We determine the fuzzy cognitive map (the network structure) of criteria, see Figure 4 b), and adjacency matrix E, shown in Figure 3 a).

Step 3: Matrix E is normalized into a matrix E^*, see Figure 3 b).

Step 4: We find the limiting matrix M using (1), the result is presented in Figure 4 a). Weights of criteria emerging from their dependencies are in each column of M, in our case m = (0.273, 0.207, 0.190, 0.293, 0.037).

Step 5: Finally, we aggregate both weights (vectors) w and m via (2): v = (0.345, 0.245, 0.171, 0.195, 0.044).

By comparison of weights w and v it can be seen that the order of criterias' weights changed a little after interdependencies were taken into account as the criterion E (equipment) is now more important than the criterion D (design) due to its higher influence on other criteria. The order of other criteria remained unchanged.

Step 6: Using relation (3) we get Q = (0.569, 0.431), so the optimal selection is the car A.

$$S = \begin{array}{c} \\ P \\ S \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{ccccc} P & S & D & E & F \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 4 & 6 \\ 1/2 & 1 & 2 & 4 & 5 \\ 1/3 & 1/2 & 1 & 2 & 3 \\ 1/4 & 1/4 & 1/2 & 1 & 3 \\ 1/6 & 1/5 & 1/3 & 1/3 & 1 \end{bmatrix} \end{array}$$

Figure 1 The reciprocal matrix S of pair-wise comparisons of criteria importance

	P	S	D	E	F
A	3	2	1/4	1	1/2
B	1/3	1/2	4	1	2

a)

	P	S	D	E	F
A	0.75	0.67	0.2	0.5	0.33
B	0.25	0.33	0.8	0.5	0.67

b)

Figure 2 a) Pair-wise comparisons of both alternatives with regard to all criteria, b) Saaty's weights of alternatives in the form of a matrix W_{32}

$$E = \begin{matrix} & P & S & D & E & F \\ P & 0 & 0.2 & 0.4 & 0.3 & 0 \\ S & 0.5 & 0 & 0.2 & 0.2 & 0 \\ D & 0.6 & 0.3 & 0 & 0.1 & 0.2 \\ E & 0.8 & 0.8 & 0.2 & 0 & 0.2 \\ F & 0.3 & 0 & 0 & 0 & 0 \end{matrix}$$

a)

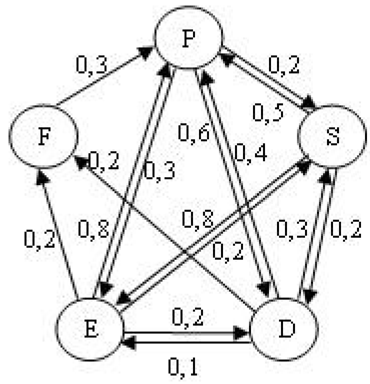
$$E^* = \begin{matrix} & P & S & D & E & F \\ P & 0 & 0.154 & 0.5 & 0.5 & 0 \\ S & 0.227 & 0 & 0.25 & 0.333 & 0 \\ D & 0.273 & 0.231 & 0 & 0.167 & 0.5 \\ E & 0.364 & 0.615 & 0.25 & 0 & 0.5 \\ F & 0.136 & 0 & 0 & 0 & 0 \end{matrix}$$

b)

Figure 3 a) The adjacency matrix *E*, b) the normalized adjacency matrix *E**

$$M = \begin{matrix} & P & S & D & E & F \\ P & 0.273 & 0.273 & 0.273 & 0.273 & 0.273 \\ S & 0.207 & 0.207 & 0.207 & 0.207 & 0.207 \\ D & 0.190 & 0.190 & 0.190 & 0.190 & 0.190 \\ E & 0.293 & 0.293 & 0.293 & 0.293 & 0.293 \\ F & 0.037 & 0.037 & 0.037 & 0.037 & 0.037 \end{matrix}$$

a)



b)

Figure 4 a) The limiting matrix *M*, b) the fuzzy cognitive map of dependence and feedback among criteria

5.2 ANP solution

The supermatrix *W* of the problem is shown in Figure 6 a), only bold blocks W_{12}, W_{22}, W_{32} and W_{33} are non-zero (see Figure 6 b)). The block W_{22} expressing the relative importance of all criteria with regard to a given criterion was obtained by Saaty's method, pair-wise comparisons are shown in Figure 5. These comparisons were based on the intensity of influence among criteria shown in the fuzzy cognitive map in Figure 4 b).

Because *W* is not stochastic (it has two blocks in one column), blocks W_{22} and W_{32} were compared for their relative importance with the result 'equal importance' (1 on Saaty's scale); hence the stochasticity of *W* was achieved by averaging in columns of blocks W_{22} and W_{32}. New stochastic supermatrix W' was then raised to powers and the limiting supermatrix W'^{∞} was found (see Figure 7). Weights (priorities) of alternatives are in the block $W_{31}'^{\infty}$: Q = (0.557, 0.443); hence the alternative A is better than B.

A comparison with the solution of HEFCM method, where Q = (0.569, 0.431), indicates there are only minor differences in alternatives' priorities, and the alternative A is evaluated better by both methods. For completeness, the solution without dependence and feedback (purely AHP) can be obtained from *W* as follows [3]: $Q = W_{32} \cdot W_{21} = (0.598, 0.402)$, which also means that A is better than B.

D	*E*	*F*	*P*	*S*
E	1	1/2	1/4	1/3
F	2	1	1/3	1/2
P	4	3	1	2
S	3	2	1/2	1

E	*D*	*F*	*P*	*S*
D	1	1	1/4	1
F	1	1	1/4	1
P	4	4	1	4
S	1	1	1/4	1

P	*D*	*E*	*S*
D	1	2	3
E	1/2	1	2
S	1/3	1/2	1

S	*D*	*E*	*P*
D	1	1/4	1/3
E	4	1	2
P	3	1/2	1

Figure 5 Pair-wise comparisons of criteria with regard to a given criterion

	G	P S D E F	A B
G	W_{11}	W_{12}	W_{13}
P S D E F	W_{21}	W_{22}	W_{23}
A B	W_{31}	W_{32}	I

a)

	G	P	S	D	E	F	A	B
G	0	0	0	0	0	0	0	0
P	0.417	0	0.320	0.467	0.571	1	0	0
S	0.282	0.163	0	0.277	0.143	0	0	0
D	0.153	0.540	0.122	0	0.143	0	0	0
E	0.098	0.297	0.559	0.095	0	0	0	0
F	0.050	0	0	0.160	0.143	0	0	0
A	0	0.75	0.667	0.2	0.5	0.333	1	0
B	0	0.25	0.333	0.8	0.5	0.667	0	0

b)

Figure 6 a) The general form of the supermatrix for dependence among criteria, b) the supermatrix of the numerical example

	G	P	S	D	E	F	A	B
G	0	0	0	0	0	0	0	0
P	0.417	0	0.160	0.234	0.286	0.5	0	0
S	0.282	0.082	0	0.139	0.072	0	0	0
D	0.153	0.270	0.061	0	0.072	0	0	0
E	0.098	0.149	0.280	0.048	0	0	0	0
F	0.050	0	0	0.080	0.072	0	0	0
A	0	0.375	0.333	0.388	0.25	0.166	1	0
B	0	0.125	0.166	0.612	0.25	0.334	0	0

a)

	G	P	S	D	E	F	A	B
G	0	0	0	0	0	0	0	0
P	0	0	0	0	0	0	0	0
S	0	0	0	0	0	0	0	0
D	0	0	0	0	0	0	0	0
E	0	0	0	0	0	0	0	0
F	0	0	0	0	0	0	0	0
A	0.557	0.607	0.601	0.388	0.528	0.470	1	0
B	0.443	0.393	0.398	0.612	0.472	0.530	0	0

b)

Figure 7 a) The stochastic matrix W', b) the limiting supermatrix W'^{∞}

6 Conclusions

In this article we proposed a new hybrid eigenvalue-fuzzy cognitive map method for modeling dependence and feedback among criteria in the ANP framework based on fuzzy cognitive maps. Our method is straightforward, computationally simple and more natural than classic ANP approach. Also, another advantage is that the method does not require the use of ad-hoc threshold functions in the process of adjacency matrix multiplication. The method was demonstrated by the example of the evaluation of criteria weights (for a purchase of a car), but it can be easily extended into cases with dependence among alternatives or among alternatives and criteria. Future research may focus on examination of similarity between HEFCM and ANP in some special cases or in general.

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