Shapley value of simple cooperative games with fuzzy coalitions applied on the real voting data

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Abstract. The main aim of this article is to compare the results of classical Shapley value concept with results of the Shapley value extended to the cooperative games with fuzzy coalitions applied on the real data of the cooperative simple game – in this case the data from the voting in the Lower House of the Czech Parliament 2002-2012. One of the most intriguing tasks is to describe real system problems using mathematical tools. As the real systems are full of an uncertainty, there are several tools incorporating the uncertainty into the classical models, for example the theory of probability, the theory of fuzzy sets, or the theory of rough sets. In this article, the probabilistic approach to the Shapley value was chosen. Results indicate the improvement in predictability of the real power of existing political coalitions comparing to the values based on the classical Shapley value.
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Keywords: Shapley value, power distribution, coalitions, Czech Parliament.

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1 Introduction

The history of the classical approach to the cooperative games evaluation goes to early 1950s, when Lloyd S. Shapley published his famous article "*A value for n-Person Games"* [4]. He presented his concept on the transferable utility cooperative games in characteristic-form function. He defined three axioms (symmetry, efficiency, and additivity) which such a value in any game should fulfill. He had shown that there exists one unique value fulfilling all three axioms, now called Shapley value.

The application of the Shapley value to the evaluation of coalitions in simple games with non-transferable utility function, and with coalitional structure – simply called voting games – was done in 1954 when L. S. Shapley and M. Shubik published their article "*A method for evaluating the distribution of power in a committee system*" [5]. They showed principle of the a-priori evaluation of power distribution in a simple game committee system.

Since then, the applications Shapley value were done in different fields, for example in economy, political science, medicine or computer science. The Shapley value was recalculated and new equivalent function form notations of the Shapley value were postulated.

Lately, as the concept of uncertainty was introduced to the field of game theory, the Shapley value was recalculated for different levels of uncertainty. For example Mareš [2] in his book "Fuzzy cooperative games" derived the Shapley value for cooperative games with fuzzy pay-offs. Butnariu and Kroupa [1] narrowed their approach to the n-person games with fuzzy coalitions under the condition of utility aggregation, Yu and Zhang [9] derived the Shapley value for fuzzy bi-cooperative games.

The main aim of this article is to compare the results of classical Shapley value concept with results of the Shapley value extended to the cooperative games with fuzzy coalitions applied on the real data of the cooperative simple game – in this case the data from the voting in the Lower House of the Czech Parliament 2002-2012. In this article, I have chosen the probabilistic approach to the Shapley value of the simple voting game.

This article is organized as follows: in the next part I will describe original idea and the basic definition of the Shapley value, as well as the basic definitions of fuzzy coalitions and the application of the classical Shapley value concept on the real data. The results of calculations and comparison of the a-priori Shapley-Shubik power index and the Shapley value of fuzzy coalitions with real voting outcomes are given in the third section. The final discussion and the list of references end the paper.

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2 Basic definitions

In the original Shapley article [4], the Shapley value is defined on the class of games in characteristic function form expressed for each subset S of universe of players U. The characteristic function v is expected to be superadditive ($v(S) \geq v(T) + v(S-T)$ for all $T, S \subseteq U$), $v(\varnothing) = 0$. Shapley defined a carrier of v to be any set $N \subseteq U$ with $v(S) = v(N \cap S)$, and $\Pi(U)$ be the set of permutations of U . The three Shapley axioms, "symmetry" (for all $\pi \in \Pi(U), \phi_{\pi i}(\pi v) = \phi_i(v)$), "efficiency" (for each carrier N of v, $\sum_N \phi_i(v) = v(N)$), and "additivity" (for all independent u, v $\phi(v+u) = \phi(v) + \phi(u)$) are sufficient to determine a unique ϕ for each player i for all games [4]:

$$\phi_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \setminus i)] \qquad (1)$$

where $N \subset U$ is any finite carrier of v. This value ϕ is called Shapley value.

The calculation of the Shapley value on the power evaluation of coalitions in simple games (games in characteristic function form v for which $v(S) \in \{0,1\}$ for all S) with non-transferable utility function is based on the existence of marginal players in coalitions [5]. Moreover, the value of players in such a game is dependent on the creation of a-priori coalitions. The calculation of these values, called Shapley-Shubik power indices together with Shapley-Shubik power indices with a-priori coalitions are described in [5, 6, 7, 8].

Let $N = \{1,2,\ldots,n\}$ denotes set of all players. By a coalition we mean any subset of *N*. Then a fuzzy coalition is a vector $A = (A(1), A(2), \ldots, A(n))$ with coordinates $A(i) \in \langle 0,1 \rangle$ called the membership degree of player i in coalition A . In the description of a voting game in the parliamentary voting, usually political parties are expected to be players of the voting game. The membership of such a player in coalition thus can be the probability of the political party occurrence in the coalition. Moreover, the characteristic function v can reach a value from the interval $\langle 0,1 \rangle$, the higher the value, the higher is the probability of the coalition to be winning. On such values, the common Shapey value (1) can be applied.

In order to evaluate the party success during a parliamentary period, the index of party success can be constructed by comparing party decision with the outcome of parliamentary voting. The party A success index is defined as the ratio of decisions of the Lower House that were the same as the party A decisions to all decisions during the parliamentary period [3]:

$$I^A_{success} = \frac{\text{number of party A decisions identical with parliamentary decisions}}{\text{number of parliamentary decisions}} \qquad (2)$$

The party decision is derived from the votes of party members using simple majority rule; the coefficient of party success is a value from the interval $\langle 0,1 \rangle$; the higher the coefficient, the higher ratio of party decisions was the same as the whole voting body decision.

3 Shapley value of real voting data

The analysis is based on the roll-call voting data from the Lower House of the Czech Parliament 2002-2012, that means two complete electoral period (2002/2006, and 2006-2010), and part of the latest electoral period (2010-2012). Data are available at the official Parliamentary site www.psp.cz. Votes of legislators are collected into voting vectors; one voting vector contains voting outcomes for one bill of all 200 members. The outcome of every vote for every member can be one element of the set {A, N, 0, Z, M}, which indicates member status or preferences: N – "no", A – "yes", Z – "present, abstain", 0 –"absent", M –"absent, excused". Every bill to be passed needs at least as many "yes" votes as the quota. The quota is based on the sum of all present legislators, which means on the sum of all legislators with the voting outcome from the set {N, A, Z}. Hence the outcome "present, abstain" in this analysis is reclassified to "no" outcome [3]. Data for the 2002-2006, 2006-2010, and 2010-2012 Lower House parliamentary periods cover 4797, 8470, and 3286 voting vectors, respectively. The last session – session 36 from the data set was on 3/14/2012. During the studied period, there were eight political parties active in the Czech Parliament; their names followed by the abbreviation used in this article are Civic Democratic Party (ODS), Czech Social Democratic Party (CSSD), Christian and Democratic Union – Czechoslovak People's Party (KDU-CSL), Czech and Moravian Communist Party (KSCM), Freedom Union (US), Green Party (SZ), TOP 09 (TOP09), Věci veřejné (VV).

A-priori Shaply-Shubik power indices

The values of an a-priori Shapley-Shubik power index are dependent on the number of legislators in political parties. The values of the Shapley-Shubik power indices for political parties present in the 2002-2006 Lower House together with Shapley-Shubik index with a-priori coalition (CSSD, KDU-CSL and US), and with the index of success are given in Table 1.The correlation coefficients of the index of success with the calculated Shapley-Shubik power index, and with the Shapley-Shubik power index with a-priori coalitions are -0.073, and 0.664, respectively. Both coefficients are not statistically significant at 0.05 levels.

	CSSD	ODS	KSCM	KDU-CSL	US
Shapley-Shubik power index	**0.400**	**0.233**	**0.233**	**0.067**	**0.067**
Shapley-Shubik power index with a-priori coalition of CSSD, KDU-CSL, and US	**0.749**	**0**	**0**	**0.125**	**0.125**
Index of Success	**0.928**	**0.596**	**0.697**	**0.904**	**0.857**

Table 1 Shapley-Shubik power indices and the index of success for political parties present in the 2002-2006 Lower House of the Czech Parliament. Source: own calculations.

Similarly, the values of the Shapley-Shubik power indices for political parties present in the 2006-2010 Lower House together with the index of success are given in Table 2.The correlation coefficient of the index of success with the calculated Shapley-Shubik power index is 0.704 and it is not statistically significant.

	CSSD	ODS	KSCM	KDU-CSL	SZ
Shapley-Shubik power index	**0.2833**	**0.3667**	**0.2833**	**0.0333**	**0.0333**
Index of Success	**0.8130**	**0.8072**	**0.7222**	**0.7485**	**0.6776**

Table 2 Shapley-Shubik power indices and the index of success for political parties present in the 2002-2006 Lower House of the Czech Parliament. Source: own calculations.

The values of the Shapley-Shubik power indices for political parties present in the 2010-2012 Lower House together with Shapley-Shubik index with a-priori coalition (CSSD, KDU-CSL and US), and with the index of success are given in Table 3.The correlation coefficients of the index of success with the calculated Shapley-Shubik power index, and with the Shapley-Shubik power index with a-priori coalitions are -0.143, and 0.843, respectively. Both correlation coefficients are not statistically significant at 0.05 levels.

	CSSD	ODS	KSCM	TOP09	VV
Shapley-Shubik power index	**0.3**	**0.3**	**0.133**	**0.133**	**0.133**
Shapley-Shubik power index with a-priori coalition of ODS, TOP09, and VV	**0**	**0.530**	**0**	**0.235**	**0.235**
Index of Success	**0.5663**	**0.9714**	**0.5426**	**0.9744**	**0.9680**

Table 3 Shapley-Shubik power indices and the index of success for political parties present in the 2010-2012 Lower House of the Czech Parliament. Source: own calculations.

Shapley value of the 2002-2010 Lower House based on real votes

In general, the Shapley value for this type of the real voting game is not a good predictor of the future success of political parties – players of the voting game. In order to study the possible predictions of future success of political parties, the Shapley values of the Lower House voting games based on (1) using real voting data (2002-2010) were calculated. Theoretical and real weights, as well as theoretical and real characteristic functions for the 2002-2006, and 2006-2010, respectively, are given in Tables 4, and 6. The same values could be calculated for the 2010-2012 parliamentary period however the period is still not over, so the values could not describe the real situation properly.

Tables contain membership degrees of all players in all possible coalitions, as well. These values are substantial for the future success prediction. Membership degrees are calculated as an average value of according votes for every coalition. Political parties are players of this voting game, thus the membership functions of players not present in coalitions might be positive.

The normalized Shapley values of political parties calculated with respect to votes in the 2002-2006 Lower House of the Czech parliament are given in Table 5. The correlation coefficients of the calculated Shapley values with the coefficient of success are 0,39, and 0.722 and are not statistically significant at 5% level.

	Theoretical weight	Theoretical v(S)	Real weight	Real v(S)	A(A)	A(B)	A(C)	A(D)	A(E)
0	0	0	0.038	0	0.141	0.080	0.069	0.072	0.085
A	0.2	0	0.067	0	0.886	0.052	0.108	0.058	0.074
B	0.2	0	0.003	0	0.081	0.820	0.086	0.283	0.267
C	0.2	0	0.073	0	0.088	0.066	0.933	0.043	0.036
D	0.2	0	0.001	0	0.240	0.205	0.146	0.658	0.158
E	0.2	0	0.004	0	0.216	0.152	0.107	0.184	0.696
AB	0.05	1	0.001	0.833	0.774	0.734	0.158	0.168	0.178
AC	0.05	0	0.136	0.158	0.947	0.031	0.895	0.029	0.033
AD	0.05	0	0.003	0	0.892	0.097	0.063	0.720	0.163
AE	0.05	0	0.009	0.045	0.857	0.049	0.089	0.156	0.807
BC	0.05	1	0.014	0.746	0.090	0.841	0.924	0.175	0.101
BD	0.05	0	0.004	0.263	0.069	0.895	0.168	0.883	0.240
BE	0.05	0	0.001	0	0.107	0.883	0.104	0.194	0.853
CD	0.05	0	0.001	0	0.085	0.241	0.810	0.768	0.210
CE	0.05	0	0.002	0.111	0.150	0.173	0.831	0.156	0.717
DE	0.05	0	0.003	0	0.165	0.187	0.052	0.798	0.868
ABC	0.0333	1	0.005	1	0.835	0.813	0.948	0.202	0.132
ABD	0.0333	1	0.004	1	0.850	0.827	0.169	0.844	0.283
ABE	0.0333	1	0.001	1	0.915	0.759	0.190	0.232	0.790
ACD	0.0333	1	0.004	0.950	0.893	0.117	0.896	0.806	0.147
ACE	0.0333	1	0.003	0.833	0.931	0.105	0.832	0.117	0.760
ADE	0.0333	0	0.008	0.175	0.910	0.159	0.076	0.825	0.899
BCD	0.0333	1	0.018	1	0.143	0.951	0.911	0.844	0.193
BCE	0.0333	1	0.005	1	0.108	0.903	0.937	0.280	0.837
BDE	0.0333	1	0.076	0.675	0.053	0.969	0.105	0.971	0.967
CDE	0.0333	0	0.002	0.400	0.226	0.266	0.842	0.832	0.822
ABCD	0.05	1	0.027	1	0.784	0.848	0.893	0.816	0.304
ABCE	0.05	1	0.006	1	0.760	0.795	0.854	0.342	0.842
ABDE	0.05	1	0.047	1	0.912	0.934	0.067	0.957	0.964
ACDE	0.05	1	0.006	1	0.919	0.180	0.914	0.717	0.733
BCDE	0.05	1	0.116	0.996	0.114	0.948	0.892	0.942	0.926
ABCDE	0.2	1	0.313	1	0.859	0.920	0.931	0.928	0.915

Table 4 Theoretical and real weights, theoretical and real characteristic functions, and membership degrees of all players in all possible coalitions for the 2002-2006 Lower House of the Czech Parliament. Codes: ODS=A, CSSD=B, KSCM=C, KDU-CSL=D, US=E. Source: own calculations.

	ODS	**CSSD**	**KSCM**	**KDU-CSL**	**US**
Normalized Shapley value	0.120	0.444	0.217	0.138	0.082
Normalized Shapley value with a-priori coalition of CSSD, KDU-CSL, and US	0.016	0.639	0.029	0.198	0.117

Table 5 Shapley value of political parties present in the 2002-2006 Lower House of the Czech Parliament based on the voting outcomes. Source: own calculations.

	Theoretical weight	Theoretical v(S)	Real weight	Real v(S)	A(A)	A(B)	A(C)	A(D)	A(E)
0	0	0	0.043	0	0.093	0.088	0.079	0.121	0.065
A	0.2	0	0.019	0.018	0.830	0.055	0.070	0.193	0.055
B	0.2	0	0.013	0.072	0.065	0.868	0.131	0.103	0.055
C	0.2	0	0.047	0	0.069	0.148	0.893	0.098	0.043
D	0.2	0	0.005	0	0.202	0.153	0.129	0.648	0.116
E	0.2	0	0.011	0	0.142	0.104	0.103	0.183	0.814
AB	0.05	1	0.009	1	0.807	0.822	0.121	0.265	0.176
AC	0.05	1	0.008	0.647	0.834	0.124	0.851	0.229	0.101
AD	0.05	0	0.020	0.186	0.906	0.069	0.060	0.800	0.137
AE	0.05	0	0.008	0.028	0.892	0.057	0.035	0.223	0.913
BC	0.05	0	0.148	0.409	0.037	0.945	0.963	0.076	0.046
BD	0.05	0	0.004	0.452	0.122	0.894	0.150	0.752	0.058
BE	0.05	0	0.003	0	0.079	0.889	0.139	0.200	0.834
CD	0.05	0	0.003	0	0.156	0.145	0.874	0.668	0.088
CE	0.05	0	0.004	0	0.152	0.132	0.858	0.204	0.883
DE	0.05	0	0.004	0	0.222	0.166	0.131	0.694	0.831
ABC	0.0333	1	0.039	1	0.778	0.834	0.869	0.306	0.169
ABD	0.0333	1	0.018	1	0.848	0.868	0.142	0.796	0.117
ABE	0.0333	1	0.007	0.983	0.844	0.855	0.126	0.332	0.912
ACD	0.0333	1	0.006	0.961	0.921	0.111	0.861	0.800	0.166
ACE	0.0333	1	0.002	0.944	0.878	0.106	0.850	0.248	0.942
ADE	0.0333	0	0.080	0.544	0.963	0.055	0.037	0.924	0.954
BCD	0.0333	1	0.019	0.949	0.108	0.943	0.965	0.777	0.087
BCE	0.0333	1	0.016	0.863	0.094	0.931	0.940	0.200	0.863
BDE	0.0333	0	0.003	0.393	0.166	0.876	0.149	0.771	0.899
CDE	0.0333	0	0.002	0.063	0.193	0.178	0.879	0.735	0.824
ABCD	0.05	1	0.087	1	0.858	0.896	0.897	0.817	0.186
ABCE	0.05	1	0.040	1	0.798	0.847	0.871	0.352	0.884
ABDE	0.05	1	0.042	1	0.931	0.852	0.107	0.902	0.957
ACDE	0.05	1	0.016	0.993	0.935	0.132	0.869	0.897	0.945
BCDE	0.05	1	0.014	0.966	0.170	0.945	0.930	0.807	0.945
ABCDE	0.2	1	0.259	1	0.907	0.912	0.921	0.879	0.935

Table 6 Theoretical and real weights, theoretical and real characteristic functions, and membership degrees of all players in all possible coalitions for the 2002-2006 Lower House of the Czech Parliament. Codes: ODS=A, CSSD=B, KSCM=C, KDU-CSL=D, SZ=E. Source: own calculations.

The normalized Shapley values of political parties calculated with respect to votes in the 2006-2010 Lower House of the Czech parliament are given in Table 7. The correlation coefficient of the Shapley values with the coefficient of success is 0,92 and is statistically significant at 5% level.

CSSD	ODS	KSCM	KDU-CSL	SZ
0.307	0.327	0.184	0.114	0.068

Table 7 Normalized Shapley value of political parties present in the 2006-2010 Lower House of the Czech Parliament based on the voting outcomes. Source: own calculations.

Expected Shapley value of the 2010-2012 Lower House

In order to predict potential success of players in voting game, we have to take into account their potential power calculated by a-priori power indices, as well as their real voting during precedent periods. In the 2010-2012 Lower House of the Czech Parliament, there were three political parties which were present in two precedent periods (ODS, CSSD and KSCM). Number of players in coalitions was determined using membership degree; the membership degree of two new players (TOP09 and VV) was estimated by the average value membership degree of all players in coalitions. The weights of coalitions were estimated as average values of weights in preceding parliamentary outcome. As players ODS, TOP09 and VV created a-priori coalition, the average weights of coalitions containing these players were adjusted to be the same as the estimated weight ratio in a-priori coalition in 2002-2006 Lower House. Estimated normalized Shapley values as well as Shapley values for a-priori coalition of ODS, TOP09 and VV are given in Table 8. The correlation coefficients are 0,849, and 0,959, respectively. Both correlation coefficients are better comparing to the a-priori Shapley-Shubik indices, even though only the second one is statistically significant at 0.05 level.

	ODS	CSSD	KSCM	KDU-CSL	US
Normalized Shapley value	0.331	0.158	0.055	0.228	0.228
Normalized Shapley value with a-priori coalition of ODS, TOP09, and VV	0.421	0	0	0.289	0.289

Table 8 Expected Normalized Shapley value of political parties present in the 2010-2012 Lower House of the Czech Parliament based on the voting outcomes in precedent periods. Source: own calculations.

4 Conclusion

This article compares the calculated a-priori Shapley-Shubik power index and the Shapley value calculated with respect to obtained votes with the voting success of players in the real voting game – in this case the data from the voting in the Lower House of the Czech Parliament 2002-2012. The calculated a-priori Shapley-Shubik power indices do not correlate with calculated indices of success. For the 2002-2006 Lower house, the correlation coefficient of the index of success with the calculated Shapley-Shubik power index is -0.073; the correlation coefficient of the index of success with the Shapley-Shubik power index with a-priori coalition structure is 0.664. The correlation coefficient of the index of success with the calculated Shapley-Shubik power index for the 2006-2010 Lower House is 0.704. All three correlation coefficients are not statistically significant at 0.05 level. The Shapley values based on real voting outcome improve the correlation – calculated correlation coefficients are higher, they vary from 0.39-0.92.

Obtained data are suitable to determine the membership degree of players – in this case political parties – in all possible coalitions. These values can be used to determine the expected values of characteristic function of the game, and can be used to estimate the real power of players. The results of such estimation evince improvement in the correlation coefficients.

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