Efficient algorithm for checking periodicity of interval circulant fuzzy matrices

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Abstract. Periodic properties of circulant interval matrices over $(\max, \min)$-algebra (fuzzy matrices) are studied. Necessary and sufficient conditions for possible and universal d-periodicity of circulant interval matrices are proved. $O(n)$ algorithm for verifying the possible d-periodicity and another $O(n \log n)$ algorithm for verifying the universal d-periodicity as well are described.

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1 Introduction

To study matrix properties in $(\max, \min)$-algebra, where addition and multiplication are formally replaced by operations of maximum and minimum, is of great importance for applications in various areas. Periodic behaviour of fuzzy matrices with corresponding polynomial algorithms were studied in [3] and [6]. However, in practice we deal often with inexact input data. This leads to demand replace scalar matrices by so-called interval matrices ([1]).

2 Preliminaries

The fuzzy algebra $\mathcal{B}$ is a triple $(B, \oplus, \otimes)$, where $(B, \leq)$ is a bounded linearly ordered set with binary operations *maximum* and *minimum*, denoted by $\oplus$, $\otimes$. The least element in B will be denoted by O, the greatest one by I.

By $\mathbb{N}$ we denote the set of all natural numbers and by $\mathbb{N}_0$ the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The greatest common divisor of a set $S \subseteq \mathbb{N}$ is denoted by $\gcd S$. For a given natural $n \in \mathbb{N}$, we use the notation N for the set of all smaller or equal positive natural numbers, i.e., $N = \{1, 2, \dots, n\}$.

For any $n \in \mathbb{N}$, $B(n, n)$ denotes the set of all square matrices of order n and $B(n)$ the set of all n-dimensional column vectors over $\mathcal{B}$. Matrix operations over $\mathcal{B}$ are defined formally in the same manner (with respect to $\oplus$, $\otimes$) as matrix operations over any other field. The rth power of a matrix A is denoted by A^r, with elements $a_{ij}^{(r)}$. For $A, C \in B(n, n)$ we write $A \leq C$ ($A < C$) if the inequality $a_{ij} \leq c_{ij}$ ($a_{ij} < c_{ij}$) holds for all $i, j \in N$.

For a matrix $A \in B(n, n)$ the symbol $G(A) = (N, E_G)$ stands for a complete, arc-weighted digraph associated with A, i.e., the node set of $G(A)$ is N, and the capacity of any arc (i, j) is a_{ij}. Let $\emptyset \neq \tilde{N} \subset N$. $G/\tilde{N}$ stands for a subdigraph of digraph $G(A) = (N, E_G)$ with the node set $\tilde{N}$ and arc set $E_{G/\tilde{N}} = \{(i, j) \in E_G;\ i, j \in \tilde{N}\}$. A path in the digraph $G(A) = (N, E_G)$ is a sequence of nodes $p = (i_1, \dots, i_{k+1})$ such that $(i_j, i_{j+1}) \in E_G$ for $j = 1, \dots, k$. The number k is the length of the path p and is denoted by $\ell(p)$. If $i_1 = i_{k+1}$, then p is called a cycle.

By a *strongly connected component* $\mathcal{K}$ of $G(A) = (N, E_G)$ we mean a subdigraph $\mathcal{K}$ generated by a non-empty subset $K \subseteq N$ such that any two distinct nodes $i, j \in K$ are contained in a common cycle

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and K is a maximal subset with this property. A strongly connected component $\mathcal{K}$ of a digraph is called non-trivial, if there is a cycle of positive length in $\mathcal{K}$. For any non-trivial strongly connected component $\mathcal{K}$ is the *period* of $\mathcal{K}$ defined as

$$\operatorname{per}\mathcal{K} = \gcd\{\,\ell(c);\ c \text{ is a cycle in } \mathcal{K},\ \ell(c) > 0\,\}.$$

If $\mathcal{K}$ is trivial, then $\operatorname{per}\mathcal{K} = 1$. By $\mathrm{SCC}^\star(G)$ we denote the set of all non-trivial strongly connected components of G. The set of all strongly connected components of G is denoted by $\mathrm{SCC}(G)$.

Definition 1. Let $A \in B(n,n)$. The *matrix period*, in notation $\operatorname{per} A$, is defined as the minimal natural number p for which there is $R \in \mathbb{N}$ such that

$$A^{k+p} = A^k \ \text{ for all } \ k \geq R.$$

In a $(\max,\min)$-algebra any element of any power of the matrix A is equal to some element of A. Therefore, the power sequence of A contains finitely many different matrices with entries of A only. As a consequence, a fuzzy matrix is always periodic in contrast to matrices in another extremal algebra, namely, max-algebra.

For given $h \in B$, the *threshold digraph* $G(A,h)$ is the digraph with the node set N and with the arc set $E_G = \{(i,j);\ i,j \in N,\ a_{ij} \geq h\}$.

The following lemma describes the relation between matrices and corresponding threshold digraphs.

Lemma 1. [4] Let A, $C \in B(n,n)$. Let h, h_1, $h_2 \in B$.

(i) If $A \leq C$ then $G(A,h) \subseteq G(C,h)$,

(ii) if $h_1 < h_2$ then $G(A,h_2) \subseteq G(A,h_1)$.

Following theorems proved in [3] are useful for study periodic properties of interval matrices.

Theorem 1. [3] *Let $A \in B(n,n)$, $d \in \mathbb{N}$. Then the following assertions are equivalent*

i) $\operatorname{per} A | d$,

ii) $(\forall h \in B)(\forall \mathcal{K} \in \mathrm{SCC}^\star(G(A,h)))\operatorname{per}\mathcal{K}|\, d$.

Theorem 2. [3] *Let $A \in B(n,n)$. Then*

$$\operatorname{per} A = \operatorname{lcm}\{\,\operatorname{per}\mathcal{K}; \mathcal{K} \in \mathrm{SCC}^\star(A)\}$$

3 Periodicity of interval circulant matrices

In this section we present a necessary and sufficient condition for an interval circulant matrix to be possibly d-periodic and a necessary and sufficient condition for an interval circulant matrix to be universally d-periodic as well. In addition we describe an $O(n)$ algorithm for verifying the possible d-periodicity and another $O(n \log n)$ algorithm for verifying the universal d-periodicity of an interval circulant matrix.

Definition 2. A matrix $A \in B(n,n)$ is called *circulant*, if it has the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_n & a_1 \end{pmatrix}.$$

We denote a circulant matrix A by abbreviation $A(a_1,\dots,a_n)$. The set of all entries of A with the same index is called *a stripe*, the entries a_k form the kth stripe. In the associated digraph $G(A)$ each stripe k defines a set of arcs of the form $(i, i+k-1)$ for $i = 1,2,\dots,n$; obviously all the numbers here are considered modulo n. Denote by $E^{(k)}$ the set of all arcs (i,j) in $G(A)$ corresponding to the kth strip.

We shall say that the *span* of an arc in $E^{(k)}$ is $s_k = k - 1$. As was observed already in [2], the arcs of $E^{(k)}$ fall into a set of disjoint cycles, all with the same length equal to 1 for the first stripe and equal to

$$\ell(c_k) = \frac{n}{\gcd(n, s_k)} \tag{1}$$

for $k \neq 1$. Denote by $C^{(k)}$ the set of all these cycles.

As a consequence results proved in lemmas below can be obtained, which allows to derive the formula for computing the period of a circulant matrix in Theorem 3.

Lemma 2. [4] Let $A(a_1, \dots, a_n)$ be a circulant matrix. For each $h \in \{a_i; i \in N\}$ the threshold digraph $G(A, h)$ is either strongly connected or consists of m strongly connected components isomorphic with $\mathcal{K}^1$, where $\mathcal{K}^1 \in \mathrm{SCC}^\star(G(A, h))$ containing node 1 and $m \mid n$.

Lemma 3. Let $G' \subseteq G$, $\mathcal{K} \in \mathrm{SCC}^\star(G)$ and $\mathcal{K}' \in \mathrm{SCC}^\star(G'/N_\mathcal{K})$. Then $\operatorname{per}\mathcal{K} | \operatorname{per}\mathcal{K}'$.

Proof. Since $\{\, \ell(c);\ c \text{ is a cycle from } \mathcal{K}' \,\} \subseteq \{\, \ell(\tilde{c});\ \tilde{c} \text{ is a cycle from } \mathcal{K} \,\}$ we obtain

$$\operatorname{per}\mathcal{K} = \gcd\{\, \ell(c);\ c \text{ is a cycle from } \mathcal{K} \,\} | \gcd\{\, \ell(\tilde{c});\ \tilde{c} \text{ is a cycle from } \mathcal{K}' \,\} = \operatorname{per}\mathcal{K}'.$$

□

Theorem 3. *Let $A(a_1, \dots, a_n)$ be a circulant matrix. Then for the period of matrix A holds*

$$\operatorname{per} A = \operatorname{per}\mathcal{K}^1$$

where $\mathcal{K}^1 \in \mathrm{SCC}^\star(G(A, \max_{i \in N} a_i))$ containing node 1.

Proof. According to Lemma 2, Lemma 3 and $G(A, \max_{i \in N} a_i) \subseteq G(A, h)$ for each $h \in \{a_k;\ k \in N\}$ we get $\operatorname{per}\mathcal{K} \mid \operatorname{per}\mathcal{K}^1$ for each $\mathcal{K} \in \mathrm{SCC}^\star(A)$. Since $\mathcal{K}^1 \in \mathrm{SCC}^\star(A)$, we have

$$\operatorname{per} A \;=\; \operatorname{lcm}\{\, \operatorname{per}\mathcal{K}; \mathcal{K} \in \mathrm{SCC}^\star(A)\} = \operatorname{per}\mathcal{K}^1.$$

□

In this paper we shall deal with matrices with interval elements. Similarly to [1], [4], [5] we define an interval matrix $\mathbf{A}$ as follows.

Definition 3. Let $\underline{A}, \overline{A} \in B(n, n)$, $\underline{A} \leq \overline{A}$. *An interval matrix* $\mathbf{A}$ with bounds $\underline{A}$ and $\overline{A}$ is defined as follows

$$\mathbf{A} = [\underline{A}, \overline{A}] = \left\{ A \in B(n, n);\ \underline{A} \leq A \leq \overline{A} \right\}.$$

Definition 4. For an interval matrix $\mathbf{A}$ of the form

$$\mathbf{A} = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \dots & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n \\ \boldsymbol{a}_n & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_{n-2} & \boldsymbol{a}_{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \dots & \boldsymbol{a}_n & \boldsymbol{a}_1 \end{pmatrix},$$

by abbreviation $\mathbf{A}(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$, where $\boldsymbol{a}_i = [\underline{a}_i, \overline{a}_i]$, $\underline{a}_i \leq \overline{a}_i$ for each $i \in N$ we define *the interval circulant matrix* $\mathbf{A}^C(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ as the set of all circulant matrices from $\mathbf{A}$, in notation $\mathbf{A}^C = \{A \in \mathbf{A}; A \text{ is circulant}\}$.

Notice that there exist matrices $A \in \mathbf{A}$ which are not circulant. Since $\underline{A}, \overline{A} \in \mathbf{A}^C$, the set $\mathbf{A}^C$ is non-empty.

Definition 5. Let d be a positive natural number. An interval circulant matrix $\mathbf{A}^C(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ is called

- *possibly d-periodic* if there exists a matrix $A \in \mathbf{A}^C$ such that $\operatorname{per} A | d$,
- *universally d-periodic* if for each matrix $A \in \mathbf{A}^C$ $\operatorname{per} A | d$ holds.

3.1 Possible d-periodicity of interval circulant matrices

Let us denote $\tilde{S} = \{\overline{a}_i;\ \max_{k\in N} \underline{a}_k \le \overline{a}_i\}$. It is clear that $\max_{k\in N} \overline{a}_k \in \tilde{S}$, so $\tilde{S} \neq \emptyset$. Let us define the number $\tilde{h} = \min \tilde{S}$ and the vector $\tilde{a} = (\tilde{a}_i) \in B(n)$ as follows:

$$\tilde{a}_i = \min\{\tilde{h}, \overline{a}_i\} \tag{2}$$

for each $i \in N$. For a given vector $a \in \boldsymbol{a}$ let us denote $h^{(a)} = \max_{i\in N} a_i$ and $J(a) = \{i \in N; a_i = h^{(a)}\}$. It is easy to see that $h^{(\tilde{a})} = \tilde{h}$ and $J(\tilde{a}) = \{i \in N; \overline{a}_i \ge \tilde{h}\}$.

The following lemma creates the base for the proof of the necessary and sufficient condition formulated in Theorem 4.

Lemma 4. Let $a \in \boldsymbol{a}$ be arbitrary and $\tilde{a}$ be given by (2). Then $J(a) \subseteq J(\tilde{a})$.

Proof. If $J(\tilde{a}) = N$, then $J(a) \subseteq J(\tilde{a})$ trivially holds. If $J(\tilde{a}) \neq N$ then it follows from (2) that for each $i \notin J(\tilde{a})$ the inequality $\overline{a}_i < \tilde{h}$ hods true. Consequently $\max_{i\notin J(\tilde{a})} \overline{a}_i \notin \tilde{S}$, i.e., $\max_{k\in N} \underline{a}_k > \max_{i\notin J(\tilde{a})} \overline{a}_i$. Let $a \in \boldsymbol{a}$, $r \in J(a)$ be arbitrary. We get

$$a_r = \max_{k\in N} a_k \ge \max_{k\in N} \underline{a}_k > \max_{i\notin J(\tilde{a})} \overline{a}_i \ge \max_{i\notin J(\tilde{a})} a_i$$

which implies $r \in J(\tilde{a})$. Consequently $J(a) \subseteq J(\tilde{a})$ for each $a \in \boldsymbol{a}$. □

The above constructed vector $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ defines a matrix $\tilde{A}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \in \mathbf{A}^C$, which plays a crucial role in checking the possible d-periodicity of the interval circulant matrix $\mathbf{A}^C$.

Theorem 4. *An interval circulant matrix $\mathbf{A}^C(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is possibly d-periodic if and only if* $\operatorname{per} \tilde{A} \mid d$.

Proof. If $\tilde{A}$ is not d-periodic then $\operatorname{per} \tilde{\mathcal{K}}^1 \nmid d$ where $\tilde{\mathcal{K}}^1 \in \mathrm{SCC}^\star(G(\tilde{A}, \tilde{h}))$ containing node 1. Let $A(a_1, a_2, \ldots, a_n)$ be arbitrary. It follows from Lemma 4 that $G(A, h^{(a)}) \subseteq G(\tilde{A}, \tilde{h})$. Then $\operatorname{per} \tilde{\mathcal{K}}^1 \mid \operatorname{per} \mathcal{K}^1$ where $\mathcal{K}^1 \in \mathrm{SCC}^\star(G(A, h^{(a)}))$ which implies $\operatorname{per} \mathcal{K}^1 \nmid d$. By Theorem 3 the matrix A is not d-periodic. This means that no matrix $A \in \mathbf{A}^C$ is d-periodic. Thus $\mathbf{A}^C$ is not possibly d-periodic.

The converse implication is trivial. □

The theorem bellow was proved in [3]. We will use the formula to compute the period of a circulant matrix in our algorithms. Denote $I(A) = \{i; a_{i+1} = \max_{k\in N} a_k\} \cup \{n\}$.

Theorem 5. [3] *Let $A(a_0, \ldots, a_{n-1})$ be a circulant matrix, let $I(A) = \{n, i_0, i_1, \ldots, i_{k-1}\}$, $|I(A)| = k+1$. Then*

$$\operatorname{per} A = \gcd\left(\frac{n}{\gcd(n, i_0)}, \frac{i_0 - i_1}{\gcd(i_0, i_1)}, \frac{i_1 - i_2}{\gcd(i_1, i_2)}, \ldots, \frac{i_{k-1} - i_{k-2}}{\gcd(i_{k-2}, i_{k-1})}\right). \tag{3}$$

Now, we can describe an algorithm based on Theorem 4 for checking the possible d-periodicity of an interval circulant matrix $\mathbf{A}^C$.

Algorithm **Possible d-periodicity of circulant matrix**

Input. $\mathbf{A} = [\underline{A}, \overline{A}]$ and d.
Output. 'yes' in variable *pdp* if $\mathbf{A}^C$ is possibly d-periodic; 'no' in *pdp* otherwise.
begin

(i) Compute $\tilde{S} = \{\overline{a}_i;\ \max_{k\in N} \underline{a}_k \le \overline{a}_i\}$;
(ii) Compute $\tilde{h} = \min \tilde{S}$;
(iii) Compute the matrix $\tilde{A}$ by $\tilde{a}_i = \min\{\tilde{h}, \overline{a}_i\}$;
(iv) Compute the period $\operatorname{per} \tilde{A}$ (by (3));
(v) **If** $\operatorname{per} \tilde{A} \mid d$ **then** *pdp* :='yes'; **else** *pdp* :='no';

end

Theorem 6. *Let $\mathbf{A}^C(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ be an interval circulant matrix. The Algorithm* **Possible d-periodicity of circulant matrix** *correctly decides in $O(n)$ time whether the interval circulant matrix $\mathbf{A}^C$ is possibly d-periodic.*

Proof. It was proved in [3] that the period of a circulant matrix can be computed in $O(n)$ time. Since none of operations in the Algorithm requires more time the total time of computation is $O(n)$. □

Let us consider the following interval matrix to illustrate the above Algorithm.

Example 1. Let $d = 4$. Let $\mathbf{A}([0,2],[2,5],[3,3],3,4],[1,3],[5,6])$ be an interval matrix.

$\tilde{S} = \{\overline{a}_i; \max_{k\in N} \underline{a}_k \leq \overline{a}_i\} = \{5,6\}$ and $\tilde{h} = \min \tilde{S} = 5$. Now, the matrix $\tilde{A}$ can be found by $\tilde{a}_i = \min\{\tilde{h}, \overline{a}_i\}$. Hence the resulting circulant matrix is $\tilde{A}(2,5,3,4,3,5)$. $I(A) = \{i; a_i = \max_{k\in N} a_k\} \cup \{n\} = \{1,5,6\}$ and now, the period $\operatorname{per} \tilde{A}$ can be computed by (3):

$$\operatorname{per} \tilde{A} = \gcd\left(\frac{6}{\gcd(6,1)}, \frac{-4}{\gcd(1,5)}\right) = \gcd(6,-4) = 2.$$

Since $\operatorname{per} \tilde{A} \mid d$ the interval circulant matrix $\mathbf{A}^C$ is possibly d-periodic.

3.2 Universal d-periodicity of interval circulant matrices

The necessary and sufficient condition for universal d-periodicity of an interval circulant matrix is formulated in the following theorem.

Theorem 7. *An interval circulant matrix $\mathbf{A}^C(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is universally d-periodic if and only if $\operatorname{per} \underline{A} \mid d$ and $(\forall k)(\overline{a}_k > \max_{i\in N} \underline{a}_i \Rightarrow \ell(c_k^1) \mid d)$, where $c_k^1 \in C^{(k)}$ containing node 1.*

Proof. Suppose that $\operatorname{per} \underline{A} \nmid d$ or there exists $k \in N$ such that $\overline{a}_k > \max_{i\in N} \underline{a}_i$ and $\ell(c_k^1) \nmid d$. We shall prove that there exists $A \in \mathbf{A}^C$ such that $\operatorname{per} A \nmid d$.

Since $\underline{A} \in \mathbf{A}^C$, in the first case there exists $A \in \mathbf{A}$ such that $\operatorname{per} A \nmid d$.

In the second case we construct the matrix $\tilde{A} = (\tilde{a}_i)$ as follows:

$$\tilde{a}_i = \begin{cases} \overline{a}_i, & \text{if } i = k, \\ \underline{a}_i, & \text{otherwise}, \end{cases}$$

where $k \in N$ is such that $\overline{a}_k > \max_{i\in N} \underline{a}_i$ and $\ell(c_k^1) \nmid d$. Since $\mathcal{K}^1 \in \mathrm{SCC}^\star(G(\tilde{A}, \overline{a}_k))$ containing node 1, consists of only cycle c_k^1 we have $\operatorname{per} \mathcal{K}^1 = \ell(c_k^1) \nmid d$. Thus $\operatorname{per} \tilde{A} \nmid d$ by Theorem 3.

For the converse implication suppose that there exists $A \in \mathbf{A}^C$ such that $\operatorname{per} A \nmid d$ and $\operatorname{per} \underline{A} \mid d$. By Theorem 3 we get $\operatorname{per} \mathcal{K}^1 \nmid d$, where $\mathcal{K}^1 \in \mathrm{SCC}^\star(G(A, \max_{i\in N} a_i))$. As $\operatorname{per} \underline{A} \mid d$, we get $\max_{i\in N} a_i > \max_{i\in N} \underline{a}_i$. Let $k \in N$ be such that $a_k = \max_{i\in N} a_i$. Since A is circulant, $c_k^1 \in \mathrm{SCC}^\star(G(A, \max_{i\in N} a_i))$. Thus there exists $k \in N$ such that $\overline{a}_k \geq a_k > \max_{i\in N} \underline{a}_i$ and $\ell(c_k) \nmid d$. □

Now, we can describe an algorithm based on Theorem 7 for checking the universal d-periodicity of an interval circulant matrix $\mathbf{A}^C$.

Algorithm **Universal d-periodicity of circulant matrix**

Input. $\mathbf{A} = [\underline{A}, \overline{A}]$ and d.
Output. 'yes' in variable *udp* if $\mathbf{A}^C$ is universally d-periodic; 'no' in *udp* otherwise.

begin

(i) $k := 0$;

(ii) Compute per $\underline{A}$ (by (3));
(iii) **If** per $\underline{A} \nmid d$ **then** *udp* :='no'; **go to** end;
(iv) Compute $\underline{a} = \max_{i \in N} \underline{a}_i$;
(v) $k := k + 1$;
(vi) **If** $k > n$ **then** *udp* :='yes'; **go to** end;
(vii) **If** $\overline{a}_k \leq \underline{a}$ **go to** (v);
(viii) Compute $l(c_k)$ (by (1));
(ix) **If** $l(c_k) \nmid d$ **then** *udp* :='no'; **go to** end; **else go to** (v);
end

Theorem 8. *Let $\mathbf{A}^C(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ be an interval circulant matrix. The Algorithm* **Universal d-periodicity of circulant matrix** *correctly decides in $O(n \log n)$ time whether the interval circulant matrix $\mathbf{A}^C$ is universally d-periodic.*

Proof. The computational complexity of the period of a circulant matrix by (3) is $O(n)$ ([3]). Therefore the total time in steps (i)-(iv) is $O(n)$. To evaluate the length of a cycle by (1) requires $O(\log n)$ operations and this will be repeated n-times. Hence the complexity of the complete algorithm is $O(n \log n)$. □

Let us consider the following interval matrix to illustrate the above Algorithm.

Example 2. Let $d = 4$. Let $\mathbf{A}([0, 4], [1, 3], [4, 6], [0, 2], [2, 3], [3, 5], [4, 4], [2, 4])$ be an interval matrix.

For the circulant matrix $\underline{A} = (0, 1, 4, 0, 2, 3, 4, 2)$ we find by (3) the period per $\underline{A} = 2$. Since per $\underline{A} \mid d$ we proceed to the next step and compute $\underline{a} = \max_{i \in N} \underline{a}_i = 4$. There are only two indices satisfying $\overline{a}_k > \underline{a}$ for which the computation of cycle length $l(c_k)$ is needed. For $k = 3$ is $l(c_k) = 4$ hence $l(c_k) \mid d$, while for $k = 6$ is $l(c_k) = 8$ and $l(c_k) \nmid d$. Thus there is a matrix $A \in \mathbf{A}^C$ with per A equal to 8. Consequently the considered interval circulant matrix $\mathbf{A}^C$ is not universally d-periodic, for $d = 4$.

A slightly modified interval matrix ($\boldsymbol{a}_5$ and $\boldsymbol{a}_6$ replace each other) of the matrix in previous example results in an universally d-periodic interval circulant matrix.

Example 3. Let $d = 4$. Let $\mathbf{A}([0, 4], [1, 3], [4, 6], [0, 2], [3, 5], [2, 3], [4, 4], [2, 4])$ be an interval matrix.

Instead of index $k = 6$ we shall consider index $k = 5$ for which the inequality $\overline{a}_k > \underline{a}$ is satisfied. Since $l(c_k) = 2$ divides $d = 4$, the given interval circulant matrix is universally d-periodic, for $d = 4$.

4 Conclusion

Polynomial algorithm for checking the possible d-periodicity with essentially improved computational complexity compared with interval matrices in general ([5]) was presented. Moreover, another polynomial algorithm for verifying the universal d-periodicity of interval fuzzy matrices was described while the computational complexity of corresponding procedure can be exponentially large in general.

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