Periodicity of interval matrices in fuzzy algebra

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Abstract. The fuzzy algebra $\mathcal{B}$ is the triple $(B, \oplus, \otimes)$, where $(B, \leq)$ is a bounded linearly ordered set with binary operations *maximum* and *minimum*. The complete describing of d-periodic interval matrices over fuzzy algebra (fuzzy matrices) is presented and d-periodicity properties are proved. Characterization of the d-periodicity of interval fuzzy matrices is described and an $O(n^5)$ algorithm for checking the possible d-periodicity of interval fuzzy matrices is suggested.

Keywords: fuzzy algebra, periodicity, interval matrix, algorithm

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1 Background of the problem

Fuzzy discrete dynamic systems can be introduced by fuzzy matrices and are useful for describing diagnosis of technical devices [9], medical diagnosis [8] or fuzzy logic programs [3]. The aim of this paper is to describe so called d-periodicity of matrices with inexact data (interval matrices) and to find algorithms for verifying the corresponding properties of interval matrices.
The fuzzy algebra $\mathcal{B}$ is the triple $(B, \oplus, \otimes)$, where $(B, \leq)$ is a bounded linearly ordered set with binary operations *maximum* and *minimum*, denoted by $\oplus$, $\otimes$.

The least element in B will be denoted by O, the greatest one by I.

By $\mathbb{N}$ we denote the set of all natural numbers and by $\mathbb{N}_0$ the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The greatest common divisor of a set $S \subseteq \mathbb{N}$ is denoted by $\gcd S$. For a given natural $n \in \mathbb{N}$, we use the notation N for the set of all smaller or equal positive natural numbers, i.e., $N = \{1, 2, \ldots, n\}$.

For any $n \in \mathbb{N}$, $B(n, n)$ denotes the set of all square matrices of order n and $B(n)$ the set of all n-dimensional column vectors over $\mathcal{B}$. The matrix operations over $\mathcal{B}$ are defined formally in the same manner (with respect to $\oplus$, $\otimes$) as matrix operations over any field. The rth power of a matrix A is denoted by A^r, with elements $a_{ij}^{(r)}$.

For A, $C \in B(n, n)$ we write $A \leq C$ $(A < C)$ if $a_{ij} \leq c_{ij}$ $(a_{ij} < c_{ij})$ holds for all i, $j \in N$.

For a matrix $A \in B(n, n)$ the symbol $G(A) = (N, E_G)$ stands for a complete, arc-weighted digraph associated with A, i.e., the node set of $G(A)$ is N, and the capacity of any arc (i, j) is a_{ij}. Let $\emptyset \neq \tilde{N} \subset N$. $G/\tilde{N}$ stands for the subdigraph of digraph $G(A) = (N, E_G)$ with the node set $\tilde{N}$ and arc set $E_{G/\tilde{N}} = \{(i, j) \in E_G;\ i, j \in \tilde{N}\}$. A path in the digraph $G(A) = (N, E_G)$ is a sequence of nodes $p = (i_1, \ldots, i_{k+1})$ such that $(i_j, i_{j+1}) \in E_G$ for $j = 1, \ldots, k$. The number k is the length of the path p and is denoted by $\ell(p)$. If $i_1 = i_{k+1}$, then p is called a cycle. A digraph $G = (N, E_G)$ without cycles is called *acyclic*. If $G = (N, E_G)$ contains at least one cycle G is called *cyclic*.

By a *strongly connected component* $\mathcal{K}$ of $G(A, h) = (N, E_G)$ we mean a subdigraph $\mathcal{K}$ generated by a non-empty subset $K \subseteq N$ such that any two distinct nodes $i, j \in K$ are contained in a common cycle and K is a maximal subset with this property. A strongly connected component $\mathcal{K}$ of a digraph is called non-trivial, if there is a cycle of positive length in $\mathcal{K}$. For any non-trivial strongly connected component

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$\mathcal{K}$ is the *period* of $\mathcal{K}$ defined as

$$\operatorname{per}\mathcal{K} = \gcd\{\,\ell(c);\ c \text{ is a cycle in } \mathcal{K},\ \ell(c) > 0\,\}.$$

If $\mathcal{K}$ is trivial, then $\operatorname{per}\mathcal{K} = 1$. By $\mathrm{SCC}^\star(G)$ we denote the set of all non-trivial strongly connected components of G. The set of all strongly connected components of G is denoted by $\mathrm{SCC}(G)$.

Definition 1. Let $A \in B(n,n)$. The *matrix period*, in notation: $\operatorname{per} A$ is defined as the minimal natural number p for which there is $R \in \mathbb{N}$ such that

$$A^{k+p} = A^k \quad \text{for all} \quad k \geq R.$$

By linearity of B, any element of any power of the matrix A is equal to some element of A. Therefore, the sequence of powers of A contains only finitely many different matrices with entries of A. As a consequence, the period of A is always well defined.

For given $h \in B$, the *threshold digraph* $G(A,h)$ is the digraph with the node set N and with the arc sets $E_G = \{(i,j);\ i,j \in N,\ a_{ij} \geq \mathrm{h}\}$.

Theorem 1. [4] *Let A, $C \in B(n,n)$. Let h, h_1, $h_2 \in B$.*

(i) If $A \leq C$ then $G(A,h) \subseteq G(C,h)$,

(ii) if $h_1 < h_2$ then $G(A,h_2) \subseteq G(A,h_1)$.

Theorem 2. [2] *Let $A \in B(n,n)$, $d \in \mathbb{N}$. Then*

i) $\operatorname{per} A | d \Leftrightarrow (\forall h \in B)(\forall \mathcal{K} \in \mathrm{SCC}^\star(G(A,h)))\operatorname{per}\mathcal{K} |\, d$,

ii) $\operatorname{per} A = \operatorname{lcm}\{\operatorname{per}\mathcal{K}; \mathcal{K} \in \mathrm{SCC}^\star(A)\}$.

2 Periodicity of interval matrices

In this section we shall deal with matrices with interval elements. Similarly to [6], [7] we define an interval matrix $\mathbf{A}$.

Definition 2. Let $\underline{A}, \overline{A} \in B(n,n)$, $\underline{A} \leq \overline{A}$. An interval matrix $\mathbf{A}$ with bounds $\underline{A}$ and $\overline{A}$ is defined as follows

$$\mathbf{A} = [\underline{A}, \overline{A}] = \left\{ A \in B(n,n);\ \underline{A} \leq A \leq \overline{A} \right\}.$$

Definition 3. An interval matrix $\mathbf{A}$ is called

- *possibly d-periodic* if there exists matrix $A \in \mathbf{A}$ such that $\operatorname{per} A | d$,
- *universally d-periodic* if for each matrix $A \in \mathbf{A}$ $\operatorname{per} A | d$ holds.

2.1 Possible d-periodicity

In this part we will prove a sufficient and necessary condition for an interval matrix to be possibly d-periodic. In addition we introduce a polynomial algorithm to check the possible d-periodicity and find the matrix $A \in \mathbf{A}$ such that $\operatorname{per} A | d$ in positive case.

Theorem 3. *Let $G' \subseteq G$, $\mathcal{K} \in \mathrm{SCC}^\star(G)$ and $\mathcal{K}' \in \mathrm{SCC}^\star(G'/N_{\mathcal{K}})$. Then* $\operatorname{per}\mathcal{K} | \operatorname{per}\mathcal{K}'$.

Proof. Since $\{\,\ell(c);\ c \text{ is a cycle from } \mathcal{K}'\,\} \subseteq \{\,\ell(\tilde{c});\ \tilde{c} \text{ is a cycle from } \mathcal{K}\,\}$ we have $\operatorname{per}\mathcal{K} = \gcd\{\,\ell(c);\ c \text{ is a cycle from } \mathcal{K}\,\} | \gcd\{\,\ell(\tilde{c});\ \tilde{c} \text{ is a cycle from } \mathcal{K}'\,\} = \operatorname{per}\mathcal{K}'$. □

Denote $H = \{\overline{a}_{ij}; i,j \in N\} = \{h^{(1)}, h^{(2)}, \ldots, h^{(r)}\}$ where $h^{(1)} > h^{(2)} > \cdots > h^{(r)}$.

Theorem 4. *An interval matrix $\mathbf{A}$ is possibly d-periodic if and only if for each $h \in H$ and for each $\mathcal{K} \in \mathrm{SCC}^\star(G(\overline{A},h))$ such that $\operatorname{per}\mathcal{K} \nmid d$ the digraph $G(\underline{A},h)/N_{\mathcal{K}}$ is acyclic.*

Proof. Suppose that there exist $h \in H$ and $\mathcal{K} \in \mathrm{SCC}^\star(G(\overline{A},h))$ such that $\operatorname{per}\mathcal{K} \nmid d$ and the digraph $G(\underline{A},h)/N_\mathcal{K} \subseteq \mathcal{K}$ contains a cycle c. Let $A \in \mathbf{A}$ be arbitrary but fixed. As $G(\underline{A},h) \subseteq G(A,h)$, there exists $\mathcal{K}^{'} \in \mathrm{SCC}^\star(G(A,h))$ such that $c \in \mathcal{K}^{'}$.

Since $G(A,h) \subseteq G(\overline{A},h)$ and $\mathcal{K}' \in \mathrm{SCC}^\star(G(A,h)/N_\mathcal{K})$ by Theorem 2 we get $\operatorname{per}\mathcal{K} \mid \operatorname{per}\mathcal{K}'$ which implies $\operatorname{per}\mathcal{K}' \nmid d$. By Theorem 2 we get $\operatorname{per} A \nmid d$. Consequently the interval matrix $\mathbf{A}$ is not possibly d-periodic.

For the converse implication suppose that the digraph $G(\underline{A},h)/N_\mathcal{K}$ is acyclic for each $h \in H$ and for each $\mathcal{K} \in \mathrm{SCC}^\star(G(\overline{A},h))$ such that $\operatorname{per}\mathcal{K} \nmid d$. We shall construct a matrix $A^* \in \mathbf{A}$ such that $\operatorname{per} A^* \mid d$. First, wel construct an axillary sequence of matrices $\{A^{(k)}\}_{k=0}^r = \{(a_{ij}^{(k)})\}_{k=0}^r$ recurrently as follows:

$$a_{ij}^{(0)} = \underline{a}_{ij} \text{ for each } i,j \in N, \tag{1}$$

$$a_{ij}^{(k+1)} = \begin{cases} h^{(k+1)} & \text{if } (i,j) \in \bigcup\limits_{s \in M} E_{\mathcal{K}^s} \text{ and } a_{ij}^{(k)} < h^{(k+1)}, \\ a_{ij}^{(k)} & \text{otherwise,} \end{cases} \tag{2}$$

for each $k \in \mathbb{N}_0$, where $\mathcal{K}^1, \mathcal{K}^2, \dots, \mathcal{K}^m \in \mathrm{SCC}^\star(G(\overline{A},h^{(k+1)}))$ are such that $\operatorname{per}\mathcal{K}^s | d$ for $s = 1,2,\dots,m$. To finish the proof, we need the following claim.

Claim. Let for each $h \in H$ and for each $\mathcal{K} \in \mathrm{SCC}^\star(G(\overline{A},h))$ such that $\operatorname{per}\mathcal{K} \nmid d$ the digraph $G(\underline{A},h)/N_\mathcal{K}$ be acyclic. Then for each $k,l \in \mathbb{N}$ such that $l \le k \le r$ holds $\operatorname{per}\mathcal{K}' \mid d$ for each $\mathcal{K}' \in \mathrm{SCC}^\star(G(A^{(k)},h^{(l)}))$.

Proof of the Claim. By mathematical induction on k

(i) For $k = 1$ we prove that $\operatorname{per}\mathcal{K}' \mid d$ for each $\mathcal{K}' \in \mathrm{SCC}^\star(G(A^{(1)},h^{(1)}))$.
Let us denote by $\mathcal{K}^1,\mathcal{K}^2,\dots,\mathcal{K}^m$ the non-trivial strongly connected components of $G(\overline{A},h^{(1)})$ such that $\operatorname{per}\mathcal{K}^s \mid d$, $s = 1,2,\dots,m$. By (2), $\mathcal{K}^s \in \mathrm{SCC}^\star(G(A^{(1)},h^{(1)}))$ for $s \le m$. Moreover $\mathrm{SCC}^\star(G(A^{(1)},h^{(1)})) = \{\mathcal{K}^1,\mathcal{K}^2,\dots,\mathcal{K}^m\}$. Consequently $\operatorname{per}\mathcal{K}' \mid d$ for each $\mathcal{K}' \in \mathrm{SCC}^\star(G(A^{(1)},h^{(1)}))$.

(ii) Suppose that $\operatorname{per}\mathcal{K}' | d$ for each for each $\mathcal{K}' \in \mathrm{SCC}^\star(G(A^{(k)},h^{(l)}))$, $l \le k$. It is easy to see that the digraphs $G(A^{(k+1)},h^{(l)})$ and $G(A^{(k)},h^{(l)})$ are identical for each $l \in \mathbb{N}$, $l \le k$. Consequently, $\operatorname{per}\mathcal{K}' \mid d$ for each $\mathcal{K}' \in \mathrm{SCC}^\star(G(A^{(k+1)},h^{(l)}))$, $l \le k$. It remains to prove that $\operatorname{per}\mathcal{K}' \mid d$ for each $\mathcal{K}' \in \mathrm{SCC}^\star(G(A^{(k+1)},h^{(k+1)}))$. It follows from the fact that $\mathrm{SCC}^\star(G(A^{(k)},h^{(k+1)})) = \{\mathcal{K} \in (G(\overline{A},h^{(k+1)})); \operatorname{per}\mathcal{K} | d\}$. □

Now, we shall continue in the proof of the theorem. If we apply the assertion of the previous claim for $k = r$, we get $\operatorname{per}\mathcal{K} \mid d$ for each $\mathcal{K} \in \mathrm{SCC}^\star(G(A^{(r)},h^{(l)}))$ and $l \le r$. Let us set $A^* = A^{(r)}$. Since $G(\overline{A},h^{(r)})$ is complete we get $a_{ij}^* \ge h^{(r)}$ for each $i,j \in N$. In order to prove that $\operatorname{per} A^* \mid d$ it remains to show that $\operatorname{per}\mathcal{K}' \mid d$ for each $\mathcal{K}' \in \mathrm{SCC}^\star(G(A^*,a_{ij}^*))$, for $i,j \in N$ such that $a_{ij}^* > h^{(r)}$, $a_{ij}^* \notin H$. It is clear that $G(A^*,a_{ij}^*) = G(A^*,h^{(p)})$, where $p \in \mathbb{N}$ is such that $h^{(p+1)} < a_{ij}^* < h^{(p)}$. Thus $\operatorname{per}\mathcal{K}' \mid d$ for each $\mathcal{K}' \in \mathrm{SCC}^\star(G(A^*,a_{ij}^*))$, $i,j \in N$. According to Theorem 2 we get $\operatorname{per} A^* \mid d$. Thus the interval matrix $\mathbf{A}$ is possibly d-periodic.

We can use the obtained results to derive an algorithm for checking the possible d-periodicity of a given interval matrix $\mathbf{A} = [\underline{A},\overline{A}]$.

Algorithm **Possible d-periodicity**

Input. $\mathbf{A} = [\underline{A},\overline{A}]$.
Output. 'yes' in variable *pp* if $\mathbf{A}$ is possibly d-periodic; 'no' in *pp* otherwise.
begin

(i) Order the elements of H in such a way that $h^{(1)} > h^{(2)} > \dots > h^{(r)}$;

(ii) $k := 0$; $A^{(k)} := \underline{A}$;

(iii) Find all strongly connected components of $G(\overline{A},h^{(k+1)})$ and for each $\mathcal{K} \in \mathrm{SCC}^\star(G(\overline{A},h^{(k+1)}))$ compute $\operatorname{per}\mathcal{K}$;

(iv) **If** there exists $\mathcal{K} \in SCC(G(\overline{A}, h^{(k+1)}))$ such that $\text{per}\,\mathcal{K} \nmid d$ and the digraph $G(\underline{A}, h^{(k+1)})/N_{\mathcal{K}}$ is cyclic, **then** $pp :=' no'$; **go to** end;

(v) **Compute** $A^{(k+1)}$ (by (2));

(vi) $k := k + 1$;

(vii) **If** $k = r + 1$ **then** $pp :=' yes'$; $A^* := A^{(r)}$ **else go to** step (iii);

end

Theorem 5. *Let* $\mathbf{A}$ *be an interval matrix. The algorithm* **Possible** d**-periodicity** *correctly decides whether a matrix* $\mathbf{A}$ *is possibly d-periodic and finds d-periodic matrix* A^* *in* $O(n^5)$ *arithmetic operations.*

Proof. To determine the complexity of the algorithm, let us recall the well-known $O(n^2)$ algorithms to find all strongly connected components of a given digraph. If the strongly connected components are known we can compute the period of each of them using the $O(n^2)$ algorithm described by Balcer and Veinott in [1]. The number of operations for checking the strong connectivity and computing the periods for a given $h \in H$ is $O(n^2 r) \leq O(n^3)$. Thus, the complexity of the complete algorithm is $|H|O(n^3) \leq n^2 O(n^3) = O(n^5)$. □

Let us consider the following interval matrix to illustrate the Possible d-periodicity algorithm.

Example 2.1. Let $O = 0$, $I = 10$ and

$$\mathbf{A} = \begin{pmatrix} [0,0] & [3,4] & [0,0] & [0,5] & [0,0] \\ [2,2] & [0,0] & [2,3] & [0,0] & [0,0] \\ [2,3] & [0,0] & [0,2] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,1] & [1,2] \\ [0,0] & [0,0] & [1,1] & [2,2] & [0,0] \end{pmatrix}.$$

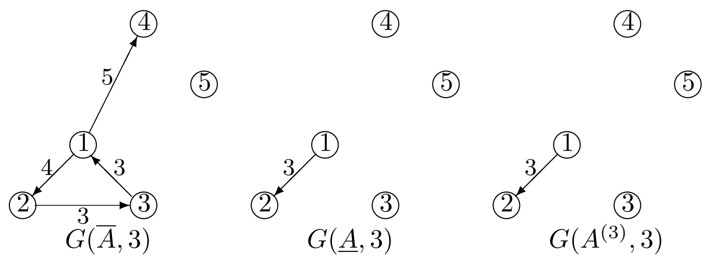

Figure 1: Threshold digraphs for $h^{(3)} = 3$

For $h^{(1)} = 5$ and $h^{(2)} = 4$ the digraphs $G(\overline{A}, h^{(1)})$, $G(\overline{A}, h^{(2)})$ are acyclic, so $A^{(1)} = A^{(2)} = \underline{A}$.
For $h^{(3)} = 3$ the digraphs $G(\overline{A}, h^{(3)})$ and $G(\underline{A}, h^{(3)})$ are presented on Figure 1.
We can see that $G(\overline{A}, h^{(3)})$ contains exactly one non-trivial strongly connected component with period $\text{per}\,\mathcal{K} = 3 \nmid 4$. As $G(\underline{A}, 3)/N_{\mathcal{K}}$ is acyclic, the condition of Theorem 4 is satisfied and $A^{(3)} = \underline{A}$.
For $h^{(4)} = 2$ there are two strongly connected components in $G(\overline{A}, h^{(4)})$: $\mathcal{K}_1$ with $N_{\mathcal{K}_1} = \{1, 2, 3\}$, $\text{per}\,\mathcal{K}_1 = 1$ and $\mathcal{K}_2$ with $N_{\mathcal{K}_2} = \{4, 5\}$, $\text{per}\,\mathcal{K}_2 = 2$ (see Figure 2). As $\text{per}\,\mathcal{K}_1 \mid 4$ and $\text{per}\,\mathcal{K}_2 \mid 4$ we compute the matrix $A^{(4)}$ from $A^{(3)}$ by increasing elements $a_{33}^{(3)}$ and $a_{45}^{(3)}$ to 2. On Figure 3 we can see that for $h^{(5)} = 1$ the digraph $G(\overline{A}, 1)$ is strongly connected with period equal to one, so we compute the matrix $A^{(5)}$ from $A^{(4)}$ by increasing elements $a_{14}^{(4)}$ and $a_{44}^{(4)}$ to 1. We get

$$A^{(4)} = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 2 & 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{2} \\ 0 & 0 & 1 & 2 & 0 \end{pmatrix} \text{ and } A^{(5)} = \begin{pmatrix} 0 & 3 & 0 & \mathbf{1} & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 1 & 2 & 0 \end{pmatrix}.$$

For $h^{(6)} = 0$ the digraph $G(\overline{A}, 0)$ is complete and $A^{(6)} = A^{(5)}$. Since $\text{per}\,A^{(5)} = 2 \mid 4$, the matrix $A^{(5)}$ is d-periodic and consequently the given interval matrix $\mathbf{A}$ is possibly d-periodic.

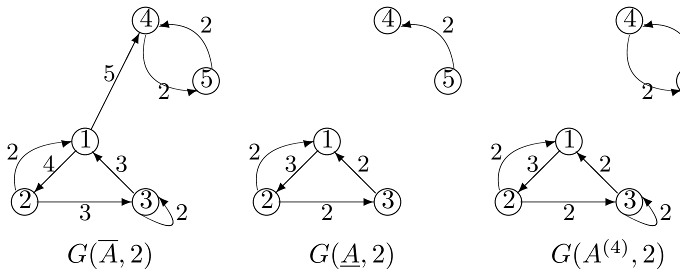

Figure 2: Threshold digraphs for $h^{(4)} = 2$

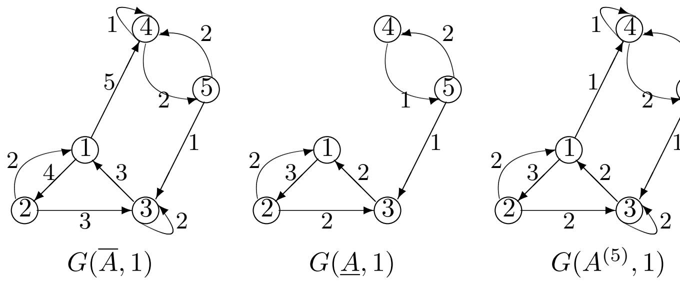

Figure 3: Threshold digraphs for $h^{(5)} = 1$

2.2 Universal d-periodicity

In this part we prove a necessary and sufficient condition for an interval matrix to be universally d-periodic.

For a given $h \in H$ let us denote $N^h = N \setminus \bigcup_{j=1}^{s} N_{\mathcal{K}^j}$, if $\mathrm{SCC}^\star(G(\underline{A}, h)) = \{\mathcal{K}^1, \ldots, \mathcal{K}^s\}$.

Theorem 6. *Let* $\mathbf{A}$ *be an interval matrix. Then* $\mathbf{A}$ *is universally* d*-periodic if and only if*

$$\operatorname{per} \underline{A} \mid d \quad \textit{and} \ (\forall h \in H)(\forall c \in G(\overline{A}, h)/N^h)\,[\ell(c) \mid d].$$

Proof. Suppose that $\operatorname{per} \underline{A} \nmid d$ or there exist $h \in H$ and $c \in G(\overline{A}, h)/N^h$ such that $\ell(c) \nmid d$.
If $\operatorname{per} \underline{A} \nmid d$ then $\mathbf{A}$ is not universally d-periodic.
In the second case we construct the matrix $\tilde{A} = (\tilde{a}_{ij})$ as follows:

$$\tilde{a}_{ij} = \begin{cases} \overline{a}_{ij}, & \text{if } \ (i,j) \in c, \\ \underline{a}_{ij}, & \text{otherwise.} \end{cases}$$

There exists $\mathcal{K}^* \in \mathrm{SCC}^\star\, G(\tilde{A}, h)$ consisting of only cycle c. Since $\operatorname{per} \mathcal{K}^* = \ell(c) \nmid d$ by Theorem 2 we have $\operatorname{per} \tilde{A} \nmid d$. Thus an interval matrix $\mathbf{A}$ is not universally d-periodic.

For the converse implication we shall suppose that $\mathbf{A}$ is not universally d-periodic and $\operatorname{per} \underline{A} \mid d$. We prove that there exist $h \in H$ and $c \in G(\overline{A}, h)/N^h$ such that $\ell(c) \nmid d$.

If $\mathbf{A}$ is not universally d-periodic then there exist $A \in \mathbf{A}$, $h \in B$ and $\mathcal{K} \in \mathrm{SCC}^\star(G(A, h))$ such that $\operatorname{per} \mathcal{K} \nmid d$. Moreover $\operatorname{per} \underline{A} \mid d$ implies $N_{\mathcal{K}} \subseteq N^h$. From $\operatorname{per} \mathcal{K} \nmid d$ it follows that there exists a cycle $c \in \mathcal{K}$ such that $\ell(c) \nmid d$. From $G(A, h)/N^h \subseteq G(\overline{A}, h)/N^h$ it follows that $c \in G(\overline{A}, h)/N^h$. Define the value $\tilde{h}$ as follows:

$$\tilde{h} = \begin{cases} h^{(r)}, & \text{if } \ h \leq \min_{i,j \in N} \overline{a}_{ij} = h^{(r)}, \\ h^{(k)}, & \text{if } \ h^{(k)} \geq h > h^{(k+1)}. \end{cases}$$

Since $N^h \subseteq N^{\tilde{h}}$ and $G(\overline{A}, h) = G(\overline{A}, \tilde{h})$ we have $c \in G(\overline{A}, \tilde{h})/N^{\tilde{h}}$. Thus there exists a cycle $c \in G(\overline{A}, \tilde{h})/N^{\tilde{h}}$ such that $\ell(c) \nmid d$. $\square$

Notice that Theorem 6 implies that the computational complexity of a procedure based on checking all cycles in $G(\overline{A}, h)/\tilde{N})$ can be exponentially large. The efficient algorithm for the interval circulant matrices is suggested in [5].

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