An algorithm for testing T5 solvability of max-plus interval systems

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Abstract. Max-plus algebra is the algebraic structure in which classical addition and multiplication are replaced by $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$, respectively. Each system of linear equation in max-plus algebra we can write in the matrix form $A \otimes x = b$, where A and b are matrix and vector of suitable size. If we replace the matrix elements with matrix interval $\boldsymbol{A} = [\underline{A}, \overline{A}]$ and vector elements by vector interval $\boldsymbol{b} = [\underline{b}, \overline{b}]$, we get an interval system of linear equations. We can define several types of solvability of interval systems in max-plus algebra. In this paper, we shall deal with one of them, the so called T5 solvability. We give the algorithm which answers the question whether the given interval system is T5 solvable or not.

Keywords: max-plus algebra, interval system, T5 solvability

JEL classification: C02
AMS classification: 15A06; 65G30

1 Introduction

Problems on algebraic structures, in which pairs of operations $(\max, +)$ or $(\max, \min)$ replace addition and multiplication of the classical linear algebra, appear in the literature approximately since the sixties of the last century (see e.g. [3, 9]). A systematic theory of such algebraic structures was published probably for the first time in [3]. One of the problems we can deal with is solving of systems of linear equations, which are useful for modeling of discrete dynamic systems, scheduling or graph theory. Among interesting real-life applications let us mention, e.g., a large scale model of Dutch railway network or synchronizing traffic lights in Delft [7].

We describe in more detail one of possible applications in the following example, taken from [7], but slightly modified and generalized.

Example 1. There are two railway stations S_1 and S_2 in a metropolitan area, which are interconnected by a railway system consisting of two inner circles and two outer circles (see Figure 1). The number $a_{ij},\ i, j \in \{1, 2\}$ indicates the transit time from station S_j to station S_i including the time necessary for passengers to change over.

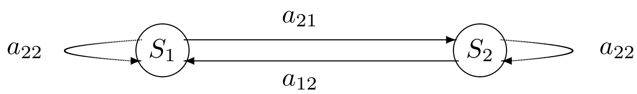

Figure 1

Suppose that there are four trains (two at each station) and two of them in the same station S_i leave simultaneously at the time x_i. The time at which both trains are already in station S_i is equal to $\max\{a_{i1} + x_1, a_{i2} + x_2\}$. Suppose that there are two schools near the two stations that begin their daily programme at the times b_1, b_2. It is required to find the departure times x_i which allow the students to catch the beginning of classes, i.e.,

$$\max_j (a_{ij} + x_j) = b_i \tag{1}$$

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for $i = 1, 2$. Using the symbols $\oplus$ and $\otimes$ for operations of maximum and addition, respectively, we can rewrite (1) to the matrix form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{2}$$

The above described model system can be easily rewritten in general case of n stations. If there is no traffic from station S_j to station S_i we set $a_{ij} = -\infty$.

2 Preliminaries

By *max-plus algebra* we understand a triple $(B, \oplus, \otimes)$, where

$$B = \mathbb{R} \cup \{\varepsilon\},\ a \oplus b = \max\{a, b\},\ a \otimes b = a + b,$$

and $\varepsilon = -\infty$. Denote by M and N the index sets $\{1, 2, \dots, m\}$ and $\{1, 2, \dots, n\}$, respectively. The set of all $m \times n$ *matrices* over B is denoted by $B(m, n)$ and the set of all column *n-vectors* over B by $B(n)$. Operations $\oplus$ and $\otimes$ are extended to matrices and vectors in the same way as in classical algebra. We shall consider the *ordering* $\leq$ on the sets $B(m, n)$ and $B(n)$ defined as follows:

- for $A, C \in B(m, n):\ A \leq C$ if $a_{ij} \leq c_{ij}$ for all $i \in M,\ j \in N$,
- for $x, y \in B(n):\ x \leq y$ if $x_j \leq y_j$ for all $j \in N$.

We shall use the *monotonicity of* $\otimes$ which means that for each $A, C \in B(m, n)$ and for each $x, y \in B(n)$ the implication

$$\text{if } A \leq C \text{ and } x \leq y \text{ then } A \otimes x \leq C \otimes y$$

holds true.

By generalizing of (2) we get a *max-plus system of linear equations* of the form

$$A \otimes x = b \tag{3}$$

where $A \in B(m, n)$, $b \in B(m)$.

To give a necessary and sufficient condition for solvability of (3), we add some conditions (for details see [6]). We shall suppose that

i) $b_i > \varepsilon$ for all $i \in M$,
ii) A contains no column with full ε-s.

By now, we can define a principal solution of system (3) as follows:

$$x^*_j(A, b) = \min_{i \in M}\{b_i - a_{ij}\} \tag{4}$$

for each $j \in N$.

The following assertions describe the importance of the principal solution for the solvability of (3).

Lemma 1. [3, 10] Let $A \in B(m, n)$ and $b \in B(m)$ be given.

i) If $A \otimes x = b$ for $x \in B(n)$, then $x \leq x^*(A, b)$.
ii) $A \otimes x^*(A, b) \leq b$.

Theorem 1. [3] *Let $A \in B(m, n)$ and $b \in B(m)$ be given. Then the system $A \otimes x = b$ is solvable if and only if $x^*(A, b)$ is its solution.*

Lemma 2. [4] Let $A \in B(m, n),\ b, d \in B(m)$ be such that $b \leq d$. Then $x^*(A, b) \leq x^*(A, d)$.

Lemma 3. [4] Let $b \in B(m),\ C, D \in B(m, n)$ be such that $D \leq C$. Then $x^*(C, b) \leq x^*(D, b)$.

3 Interval systems

In practice, the traveling times in Example 1 may depend on outside conditions, so the values a_{ij} are from intervals of possible values, i.e., $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$ for each $i \in N,\ j \in N$. Also we shall require the arrival times to be not precise values but they are rather from given intervals, i.e., $b_i \in [\underline{b}_i, \overline{b}_i]$ for each $i \in N$.

Similarly to [1, 4, 5, 8] we define an *interval matrix* $\boldsymbol{A}$ and *interval vector* $\boldsymbol{b}$ as follows:

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \left\{ A \in B(m,n);\ \underline{A} \le A \le \overline{A} \right\}$$

and

$$\boldsymbol{b} = [\underline{b}, \overline{b}] = \left\{ b \in B(m);\ \underline{b} \le b \le \overline{b} \right\},$$

where $\underline{A}, \overline{A} \in B(m,n)$, $\underline{A} \le \overline{A}$ and $\underline{b}, \overline{b} \in B(m)$, $\underline{b} \le \overline{b}$.

Denote by

$$\boldsymbol{A} \otimes x = \boldsymbol{b} \tag{5}$$

the set of all max-plus systems of linear equations of the form (3) such that $A \in \boldsymbol{A},\ b \in \boldsymbol{b}$. We shall call (5) a *max-plus interval system of linear equations.*

A system of the form (3) is called a *subsystem* of (5) if $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$. We say, that interval system (5) has the *constant matrix* if $\underline{A} = \overline{A}$ and has the *constant right-hand side*, if $\underline{b} = \overline{b}$. Subsystem (3) is *extremal*, if each of the equations has the form $[\underline{A} \otimes x]_i = \overline{b}_i$ or $[\overline{A} \otimes x]_i = \underline{b}_i$ and we call them an *LU equation* or an *UL equation*, respectively.

To use the arguments from the previous section we shall suppose for interval system (5) that

i) $\underline{b}_i \neq \varepsilon$ for each $i \in M$,
ii) for each $j \in N$ there exists $i \in M$ such that $\underline{a}_{ij} \neq \varepsilon$.

We can define several conditions which the given interval system is required to fulfill. According to them we shall define several solvability concepts. Table 1 contains the list of all up to now studied types of the solvability of (5) in max-plus algebra. There are omitted solvability concepts which lead to trivial conditions.

Solvability concept	**Definition**
Weak solvability [1]	$(\exists x \in B(n))(\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})\colon A \otimes x = b$
Strong solvability [2]	$(\forall A \in \boldsymbol{A})(\forall b \in \boldsymbol{b})(\exists x \in B(n))\colon A \otimes x = b$
Tolerance solvability [1]	$(\exists x \in B(n))(\forall A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})\colon A \otimes x = b$
Weak tolerance solvability [4]	$(\forall A \in \boldsymbol{A})(\exists x \in B(n))(\exists b \in \boldsymbol{b})\colon A \otimes x = b$
Control solvability [5]	$(\exists x \in B(n))(\forall b \in \boldsymbol{b})(\exists A \in \boldsymbol{A})\colon A \otimes x = b$
Weak control solvability [5]	$(\forall b \in \boldsymbol{b})(\exists x \in B(n))(\exists A \in \boldsymbol{A})\colon A \otimes x = b$
Universal solvability [4]	$(\exists x \in B(n))(\forall b \in \boldsymbol{b})(\forall A \in \boldsymbol{A})\colon A \otimes x = b$
Weak universal solvability [5]	$(\forall b \in \boldsymbol{b})(\exists x \in B(n))(\forall A \in \boldsymbol{A})\colon A \otimes x = b$
T4 solvability [6]	$(\exists b \in \boldsymbol{b})(\exists x \in B(n))(\forall A \in \boldsymbol{A})\colon A \otimes x = b$

Table 1: Solvability concepts

4 T5 solvability

The notions of a T5-vector and the T5 solvability of interval system (5) are defined in this section. The procedure for checking the T5 solvability is presented.

Definition 1.

i) A vector $b \in \boldsymbol{b}$ is called a T5-vector of interval system (5), if for each $A \in \boldsymbol{A}$ there exists $x \in B(n)$ such that $A \otimes x = b$.
ii) Interval system (5) is T5 solvable if there exists a vector $b \in \boldsymbol{b}$ such that b is its T5-vector.

To give a necessary and sufficient condition for the T5 solvability we recall the notion of the *strong solvability*, which has been studied in max-plus and max-min algebra by K. Cechlárová and R. Cuninghame-Green [2].

Definition 2. Interval system (5) is strongly solvable if for each $b \in \boldsymbol{b}$ and for each $A \in \boldsymbol{A}$ there exists $x \in B(n)$ such that $A \otimes x = b$.

Theorem 2. [2] *Interval system (5) is strongly solvable if and only if all its extremal subsystems with exactly one LU equation are solvable.*

For each $k = 1, 2, \ldots, m$ denote by $A^{(k)}$ the matrix with entries

$$a_{ij}^{(k)} = \begin{cases} \underline{a}_{ij} & \text{for} \quad i = k,\ j \in N, \\ \overline{a}_{ij} & \text{for} \quad i \neq k,\ j \in N. \end{cases}$$

Lemma 4. A vector $b \in \boldsymbol{b}$ is a T5-vector of interval system (5) if and only if

$$A^{(k)} \otimes x^*(A^{(k)}, b) = b \tag{6}$$

for each $k \in M$.

Proof. A vector $b \in \boldsymbol{b}$ is a T5-vector of interval system (5) if and only if interval system (5) with the constant right-hand side $\underline{b} = \overline{b} = b$ is strongly solvable. By Theorem 1 and Theorem 2, the strong solvability of an interval system (5) with $\underline{b} = \overline{b} = b$ is equivalent to the validity of (6) for each $k \in M$. □

This lemma does not give a method for finding a T5-vector. For this reason, we define a *T5-sequence* of interval system (5).

Definition 3. The T5-sequence of interval system (5) is a sequence $\{c^{(k)}\}_{k=0}^{\infty}$ defined as follows:

$$\begin{aligned} c^{(0)} &= \overline{b}, \\ c_i^{(k+1)} &= \min_{r \in M} \{[A^{(r)} \otimes x^*(A^{(r)}, c^{(k)})]_i\}, \end{aligned} \tag{7}$$

for each $k \in \mathbb{N}_0$, $i \in M$.

Theorem 3. *Let $c \in \boldsymbol{b}$ be a T5-vector of interval system (5). Then for each nonnegative integer k the inequality $c \leq c^{(k)}$ is satisfied.*

Proof. By mathematical induction on k

1. For $k = 0$ the inequality $c \leq \overline{b} = c^{(0)}$ is trivially satisfied.
2. We prove that if $c \leq c^{(k)}$ then $c \leq c^{(k+1)}$.
   For the sake of contradiction suppose that $c \leq c^{(k)}$ and there exists $i \in M$ such that $c_i > c_i^{(k+1)}$. Using Lemma 2 we get

$$c_i > \min_{r \in M} \{[A^{(r)} \otimes x^*(A^{(r)}, c^{(k)})]_i\} \geq \min_{r \in M} \{[A^{(r)} \otimes x^*(A^{(r)}, c)]_i\},$$

   which implies that there exists $p \in M$ such that $A^{(p)} \otimes x^*(A^{(p)}, c) \neq c$. By Lemma 4 the vector c is not a T5-vector of (5), a contradiction. □

Lemma 5. Let $\{c^{(k)}\}_{k=0}^{\infty}$ be the T5-sequence of interval system(5) and $l \in \mathbb{N}_0$ be arbitrary. The following assertions hold true:

i) The sequence $\{c^{(k)}\}_{k=0}^{\infty}$ is non-increasing.
ii) A vector $c^{(l)} \in \boldsymbol{b}$ is a T5-vector of (5) if and only if $c^{(l+1)} = c^{(l)}$.

Proof.

i) By Lemma 1b) we have $[A^{(r)} \otimes x^*(A^{(r)}, c^{(k)})]_i \leq c_i^{(k)}$ for each $r \in M$, $i \in M$ which implies $c_i^{(k+1)} = \min_{r \in M}\{[A^{(r)} \otimes x^*(A^{(r)}, c^{(k)})]_i\} \leq c_i^{(k)}$ for each $i \in M$, so the sequence $\{c^{(k)}\}_{k=0}^{\infty}$ is non-increasing.

ii) According to Lemma , a vector $c^{(l)}$ is a T5-vector of (5) if and only if $[A^{(r)} \otimes x^*(A^{(r)}, c^{(l)})]_i = c_i^{(l)}$ for each $r \in M$, $i \in M$ which is equivalent to $c_i^{(l+1)} = \min_{r \in M}\{[A^{(r)} \otimes x^*(A^{(r)}, c^{(l)})]_i\} = c_i^{(l)}$. □

Now, we can suggest the algorithm for checking the T5 solvability.

Algorithm T5

Input: $\boldsymbol{A}$,$\boldsymbol{b}$

Output: 'yes' in variable $t5$ if the given interval system is T5 solvable, and 'no' in $t5$ otherwise

Step 1. $c^{(0)} = \overline{b},\ k = 0$

Step 2. For each $i \in M$ compute $c_i^{(k+1)} = \min_{r \in M}\{[A^{(r)} \otimes x^*(A^{(r)}, c^{(k)})]_i\}$

Step 3. If $\underline{b} \not\leq c^{(k+1)}$ then $t5 := no$, go to **end**
Step 4. If $c^{(k+1)} = c^{(k)}$ then $t5 := yes$, $c^* = c^{(k)}$ go to **end**
Step 5. $k = k+1$, go to *Step 2*
end

Remark 1. If interval system (5) in the max-plus algebra is T5 solvable then vector c^* is its maximal T5-vector.

Remark 2. Using Algorithm T5 for the model system described in Example 1 we can find the vector of arrival times $c^* \in \boldsymbol{b}$ which can be achieved by suitable choice of the vector x of the departure times depending of the transit times between the stations, if such a vector of arrival times exists.

Example 2. Check the T5 solvability of the interval system $\boldsymbol{A} \otimes x = \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{pmatrix} [16,17] & [5,12] & [8,10] \\ [7,13] & [11,12] & [5,5] \\ [4,8] & [1,3] & [15,15] \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} [20,22] \\ [15,18] \\ [14,17] \end{pmatrix}.$$

We have

$$A^{(1)} = \begin{pmatrix} 16 & 5 & 8 \\ 13 & 12 & 5 \\ 8 & 3 & 15 \end{pmatrix}, \ A^{(2)} = \begin{pmatrix} 17 & 12 & 10 \\ 7 & 11 & 5 \\ 8 & 3 & 15 \end{pmatrix}, \ A^{(3)} = \begin{pmatrix} 17 & 12 & 10 \\ 13 & 12 & 5 \\ 4 & 1 & 15 \end{pmatrix}.$$

and $c^{(0)} = (22, 18, 17)^T$. We compute

$$x^*(A^{(1)}, c^{(0)}) = \begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix}, \quad A^{(1)} \otimes x^*(A^{(1)}, c^{(0)}) = \begin{pmatrix} 21 \\ 18 \\ 17 \end{pmatrix},$$

$$x^*(A^{(2)}, c^{(0)}) = \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix}, \quad A^{(2)} \otimes x^*(A^{(2)}, c^{(0)}) = \begin{pmatrix} 22 \\ 18 \\ 17 \end{pmatrix},$$

$$x^*(A^{(3)}, c^{(0)}) = \begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix}, \quad A^{(3)} \otimes x^*(A^{(3)}, c^{(0)}) = \begin{pmatrix} 22 \\ 18 \\ 17 \end{pmatrix}.$$

By (7), we get $c^{(1)} = (21, 18, 17)^T$ and consequently we compute

$$x^*(A^{(1)}, c^{(1)}) = \begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix}, \quad A^{(1)} \otimes x^*(A^{(1)}, c^{(1)}) = \begin{pmatrix} 21 \\ 18 \\ 17 \end{pmatrix} = c^{(1)},$$

$$x^*(A^{(2)}, c^{(1)}) = \begin{pmatrix} 4 \\ 7 \\ 2 \end{pmatrix}, \quad A^{(2)} \otimes x^*(A^{(2)}, c^{(1)}) = \begin{pmatrix} 21 \\ 18 \\ 17 \end{pmatrix} = c^{(1)},$$

$$x^*(A^{(3)}, c^{(1)}) = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}, \quad A^{(3)} \otimes x^*(A^{(3)}, c^{(1)}) = \begin{pmatrix} 21 \\ 18 \\ 17 \end{pmatrix} = c^{(1)}.$$

Since $A^{(k)} \otimes x^*(A^{(k)}, c^{(1)}) = c^{(1)}$ for $k = 1, 2, 3$, the given interval system is T5 solvable with the vector $c^{(1)} = c^*$ as the maximal T5-vector.

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