Skip pickup and delivery problem with vehicles circulation

Jan Pelikán[1]

Abstract. The skip transport consists in transport of skips (big containers, trailers) from initial location to destination location using vehicles (tractors). Capacity of vehicles is limited, usually the capacity of the vehicles is one or two containers. The problem is defined on the graph, nodes are initial and destination locations, total transport cost is minimized. It is supposed that depot is given, all vehicles start and finish in the depot. There are many papers which pay attention to skip delivery problem (SDP). It was shown that the problem SDP can be solved as b-matching problem for case of vehicle capacity two, and the case of capacity one or two. Unlike literature results the skip pickup and delivery problem with vehicle circulation is studied (SPDPC). Depots of vehicles are not given, but for each vehicle the cyclical path in the graph should be found and depot the path can be arbitrary node on this path. All cyclical paths create a circulation of the multigraph, where each arc can be multiplied. The total transport costs have to be minimized. The problem SPDPC is a special case of the pickup and delivery problem with transfers and split demand, but the transport demand, the capacity of containers and a solution of the problem are integer. A mathematical model is formulated for the case of distance matrix with triangular inequality and the capacity of vehicles one. The matrix of the model is totally unimodular then the problem is polynomially solved. A method for the the circulation of vehicles is proposed. It is demonstrated on a numerical example.

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1 Introduction - skip delivery problem

There are many papers that pay attention to skip delivery problem. In [1] the transport problem of skips is studied, in which skips are transported by vehicles with capacity one or two from depot to customers. There is a set of customers with the skip demand. Each customer is the node in the graph $G = \{V, E\}$, V is a set of nodes, E a set of edges, node 1 is depot. Demand of node i is denoted by positive integer b_i. The cost of travel from node i to node j is c_{ij}, it is supposed that the triangular inequality is satisfied and the matrix C is symmetrical. Number of vehicles in the depot is unlimited, the capacity Q of vehicles is fixed and positive integer and the depot of all vehicles is the node 1. All skips are available in the depot and from the depot are delivered to the nodes according the node demand b_i.

1.1 Skip delivery problem [1], case Q=2

In the case Q=2 there are only two possible round trips:
a) from the depot 1 to the node i and the node j and return to the node 1, this trip we denote e_{ij},
b) from the depot 1 to the node i and return to the depot, the trip is e_{ii}.

Now we define an artificial graph problem: minimum weight b-matching problem as follows. The graph $G' = \{V', E'\}$, where $V' = V - \{1\}$. The set of edges E' contains all round trips e_{ij} and e_{ii},

[1]University of Economics Prague/W. Churchill Sq.4, Prague, e-mail: pelikan@vse.cz

where trip e_{ij} is edge of the graph G' and e_{ii} the loop of the graph G'. The weight of the edge e_{ij} is $c'_{ij} = c_{1i} + c_{ij} + c_{j1}$, and the weight of the loop e_{ii} is $c_{ii} = c_{1i} + c_{i1}$. SDP problem with Q=2 can be formulated as minimal weighted b-matching problem of the graph G'. The mathematical formulation of the b-matching problem is as follows: Let us have integer variables x_{ij}, where (i, j) is edge or loop of the graph G'. The value x_{ij} for $i \neq j$ is a number of trips of the type a) and x_{ii} a number of trips of the type b). Mathematical model of the minimal weighted b-matching problem is:

$$z = \sum_{(i,j)\in E'} c'_{ij} x_{ij} \to min \tag{1}$$

$$\sum_{(i,j)\in E', i\neq j} x_{i,j} + \sum_{i\in V'} 2x_{ii} \geq b_i, i \in V' \tag{2}$$

$$x_{i,j}, \text{integer}, (i, j) \in E' \tag{3}$$

The problem is reducible, i.e. there is an optimal solution of the instance in which each node is served by as many full load depot-node trips as possible. Reducible instance can be transformed into an instance of the generalized minimum cost matching problem, because demand of nodes is at least only one container.

1.2 Skip delivery problem [1], case Q=1 or 2

Similar results hold for the case with capacity of vehicles is one or two. At first the cost of transfer thru the edge (i, j) must differ for vehicle with capacity one or capacity two. The cost of travel from node i to node j is $c'_{ij} = k_1 c_{ij}$ if the capacity of the vehicle is one, and $c'_{ij} = k_2 c_{ij}$ if the capacity is two, where c_{ij} is distance from node i to node j. If $k_2/k_1 \leq 1$ we will use only vehicle with capacity two, in case $k_2/k_1 \geq 2$ all used vehicles are with capacity one in the optimal solution. In the case Q=1 or 2 we construct the undirected complete graph $G' = \{V, E'\}$ as an instance of the minimum weight b-matching problem in the following way: $c'_{ij} = k_2(c_{1i} + c_{ij} + c_{j1}), i \neq j, i, j \in V - \{1\}$ and x_{ij} gives the number of trips $1 - i - j - 1$ with a vehicle with capacity two, $c'_{1i} = 2k_1 c_{1i}$ for $i \in V - \{1\}$ and x_{1i} is the number of trips $1 - i - 1$ and vehicle capacity one, $c'_{ii} = 2k_2 c_{1i}$ for $i \in V - \{1\}$ and x_{ii} is number of trips $1 - i - 1$ and vehicle capacity two.

Example. Given graph with 4 nodes $V = 1, 2, 3, 4$, node 1 is depot. Skip demand is $b = (0, 3, 5, 4)$, , cost matrix is $C = \begin{pmatrix} 0 & 15 & 25 & 31 \\ 15 & 0 & 19 & 27 \\ 25 & 19 & 0 & 29 \\ 31 & 27 & 29 & 0 \end{pmatrix}$.

The mathematical model of the case $Q = 2$ is:

$$z = 30x_{22} + 59x_{23} + 73x_{24} + 50x_{33} + 85x_{34} + 62x_{44} \to min$$

$$2x_{22} + x_{23} + x_{24} \geq 3$$

$$x_{23} + 2x_{33} + x_{34} \geq 5$$

$$x_{24} + x_{34} + 2x_{44} \geq 4$$

$$x_{i,j}, \text{integer}, i, j = 2, 3, 4.$$

The optimal solution is $x_{22} = 1, x_{23} = 1, x_{33} = 2$ with $z = 313$.

The mathematical model of the case $Q = 1, 2$ and $k_1 = 4/5, k_2 = 5/4$ is:

$$z = 24x_{12} + 40x_{13} + 49x_{14} + 30x_{22} + 59x_{23} + 73x_{24} + 50x_{33} + 85x_{34} + 62x_{44} \to min$$

$$x_{12} + 2x_{22} + x_{23} + x_{24} \geq 3$$

$$x_{13} + x_{23} + 2x_{33} + x_{34} \geq 5$$

$$x_{14} + x_{24} + x_{34} + 2x_{44} \geq 4$$

$$x_{i,j}, \text{integer}, i, j = 1, 2, 3, 4.$$

The optimal solution is $x_{14} = 4, x_{22} = 1, x_{23} = 1, x_{33} = 2$ with $z = 285$.

Under assumption of symmetrical cost matrix and holding triangular inequality the problem with capacity vehicles one or two is reducible.

The b-matching problem can be solved in polynomial time (see [1][2]) because the polytope of the b-matching model (1) (2) (3) with the blossom inequality (see [4]) is integral.

2 Skip Pickup and Delivery Problem

2.1 Definition of skip pickup and delivery problem

Pickup and delivery problem is defined in [5]. Given a distribution network with a set of n nodes and the cost matrix C of the travel costs between all pairs of nodes, where c_{ij} is the cost - distance between nodes i and j. Let us denote d_{kl} the number of skips that has to be transported from node k to node l. Vehicles with capacity Q are used for pickup and delivery and they can start in any node. All routes have to be cyclical, each vehicle has to come back to the node it starts from. The objective is to minimize total cost of all routes. The optimal solution is a set of cyclical routes, for each of them a depot is specified. The cyclical routes have to cover all pickup and delivery demands D. Triangular inequality and symmetry for C is supposed. The problem can be solved by the method which is shown on the example as follows.

The skip pickup and delivery problem is a special case of the pickup and delivery problem with transfers and split demand SDPDPT [3], where capacity of vehicles is Q=1. The mathematical model of SDPDPT is:

Parameters:

C	matrix of the length of arcs,
D	matrix of the goods flows between two nodes,
$Q = 1$	the capacity of the vehicle,
n	the number of nodes.

Variables:

y_{ij}	number of vehicles going thru the arc (i, j) $(i, j = 1, 2, ..., n + 1,\ i \neq j)$,
x_{ij}^{kl}	an amount of goods (a part of the total amount d_{kl}) transported from node i to node j, $(i, j = 1, 2, ..., n + 1,\ i \neq j,\ k, l = 1, 2, ..., n,\ k \neq l)$.

The SDPDPT model:

$$F(Y) = \sum_{i,j} c_{ij} y_{ij} \to min, \tag{4}$$

$$\sum_i y_{ij} - \sum_i y_{ji} = 0,\ j = 1, 2, ..., n \tag{5}$$

$$\sum_i x_{ij}^{kl} - \sum_i x_{ji}^{kl} = \left\{ \begin{array}{ll} -d_{kl}, & j = k \\ d_{kl}, & j = l \\ 0, & j \neq k, l \end{array} \right\} k, l = 1, 2, ..., n,\ k \neq l \tag{6}$$

$$\sum_{k,l} x_{ij}^{kl} \leq Q y_{ij},\ i, j = 1, 2, ..., n,\ i \neq j \tag{7}$$

$$x_{ij}^{kl} \geq 0 \text{ integer},\ \ k, l = 1, 2, ..., n,\ k \neq l,\ i, j = 1, 2, ..., n,\ i \neq j \tag{8}$$

$$y_{ij} \geq 0 \text{ integer},\ i, j = 1, 2, ..., n,\ i \neq j\,. \tag{9}$$

Comment.The model (5)-(10) is a multi-product flow problem, then the matrix of the constraints is not totally unimodular, even if $Q = 1$, and (see [4] [2]) therefore the polyhedron of the model is not integral.

Lemma 1: If the non negative costs matrix satisfies the triangular inequality then the length c_{ij} of the arc (i, j) is less or equal than the length of any path from the node i to the node j.

Proof. Let us have the path $(k_1, k_2, ..., k_s)$, where $k_1 = i$ and $k_s = j$ then $c_{ij} \leq c_{i,k_2} + c_{k_2,j} \leq c_{k_1,k_2} + c_{k_2,k_s} \leq c_{k_1,k_2} + c_{k_2,k_3} + c_{k_3,k_s} \leq \cdots \leq c_{k_1,k_2} + c_{k_2,k_3} + \cdots + c_{k_{s-1},k_s}$.

Lemma 2: Let (Y, X) holds the constraints (6)-(10) and one skip from the transport requirement d_{ij} is transported thru the path $P = (k_1, k_2, ..., k_s)$, where $k_1 = i$ and $k_s = j$. Let (Y', X') is the solution obtained from (Y, X), where the transport of this skip thru the path P is replaced by the arc (i, j). Then $F(Y) \geq F(Y')$.

Proof follows from the lemma 1.

Proposition 1. *Let the matrix C is nonnegative symmetric with triangular inequality, transport requirement matrix D is integer and the capacity of vehicles is equal one. Then the skip pickup and delivery problem can be solved in polynomial time by the model (11)-(13).*

Proof. It follows from lemma 2 that all skip transport requirements d_{ij} have to transported direct thru arc (i, j). So the solution Y has to meet the flow equation (12) to ensure the existence a set of cyclical routes and inequalities (13) which ensures all transport requirements D. The constraints matrix of the model (11)-(13) is node-arc matrix, which is totally unimodular, therefore the polytope of the model is integral, so all extremal solutions are integer. □

$$z = \sum_{(i,j)} c_{ij} y_{ij} \rightarrow min \tag{10}$$

$$\sum_{(i,j), i \neq j} y_{ij} + \sum_{(j,k), j \neq k} y_{jk} = 0,\ j = 1, 2, ..., n \tag{11}$$

$$y_{ij} \geq d_{ij},\ i, j = 1, 2, ..., n \tag{12}$$

Example. There are 5 nodes and capacity of vehicle one. Travel costs C and transport requirements Q are :

$$C = \begin{pmatrix} 0 & 3 & 7 & 3 & 2 \\ 3 & 0 & 2 & 1 & 6 \\ 7 & 2 & 0 & 2 & 3 \\ 3 & 1 & 2 & 0 & 6 \\ 4 & 6 & 3 & 6 & 0 \end{pmatrix} D = \begin{pmatrix} 0 & 0 & 0 & 0 & 4 \\ 1 & 0 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{pmatrix} \text{and the solution } Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 4 \\ 1 & 0 & 2 & 5 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 4 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \end{pmatrix} \text{is}$$

optimal.

The number of vehicles traveling thru edge (i, j) without load is $y_{32} = 1, y_{41} = 3, y_{42} = 4, y_{51} = 2$ with cost 17. The number of full loaded vehicle going thru edges is $y_{ij} = q_{ij}$ with costs 41. Total costs 41+17=58 are optimal.

2.2 Cyclical routes generation

A number of vehicles entering each node equals to a number of vehicles leaving it in the optimal solution of the model (11), (12) and (13). For generation of a set of the cyclical routes, each in the form of a path $(i_1, i_2, ..., i_t)$, the following general algorithm can be used (see [3]):

Algorithm for the route generation:
Step 1. If $y_{ij} = 0$ for all arcs (i, j), it is not possible to generate any route, otherwise select any arc $(i_1, i_2), y_{i_1,i_2} > 0$. Set $y_{i_1,i_2} = y_{i_1,i_2} - 1$ and $t = 2$.

Step 2. Repeat while $i_1 \neq i_t$: Select any arc (i_t, y_{t+1}) for $y_{i_t,i_{t+1}} > 0$. Set $y_{i_t,i_{t+1}} = y_{i_t,i_{t+1}} - 1$ and $t = t + 1$.

Example.

Using the optimal value of variables y and the algorithm for cyclical route generation we get the routes in the table 1.

cyclical route	number of routes
1-5-1	2
1-5-2-1	1
1-5-2-4-1	1
2-4-2	4
2-3-5-2	1
2-3-2	1

Table 1: Optimal routes

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