Double system parts optimization: static and dynamic model

Jan Pelikán[1], Jiří Henzler[2]

Abstract. A proposed optimization model deals with the problem of reserves for the functional components-parts of mechanism in order to increase its reliability. The following factors are taken into consideration: the probability of the failure-free run of a part without a reserve, the probability of the failure-free run of a part with a reserve, the mean value of losses caused by the part's malfunction without a reserve, the mean value of losses caused by the part's malfunction with a reserve, costs of the purchase and maintenance of the reserve for the given parts. The values of these parts' failure probabilities are supposed to be known in advance, the losses caused by this failure are estimated. Statistical independence of the failures of those parts is supposed. In the model, the costs of the parts' doubling are supposed to be limited to a fixed value. As a result of the problem solution, the parts of the model are sorted into two groups: the parts which are to be doubled and the parts which are not. The static model as well as the dynamic one, in which the failures are considered as Poisson events are described, including numerical examples.

Keywords: doubling of system parts, optimization model, probability of failure, mean value of losses, system reliability, static and dynamic model.

JEL classification: C44
AMS classification: 90C15

1 Introduction

Complex mechanisms are composed of a great number of components. Each of the parts is responsible for the right functioning of the whole system and vice versa, each part's failure can disturb the system or completely put it out of operation and cause damages in its effect.

One of the ways of eliminating or at least diminishing these damages is doubling of some important parts. Having these parts doubled, there is a possibility to exchange immediately a non-functional part by a functional one (or in other words, the failure is reduced only to a necessary time of a switch-over).

On the other side there are costs of doubling, primarily the price of a doubled part. For that reason not all of the parts can be doubled, especially the expensive ones and also those whose failure is not bringing so expensive damages.

2 Double system parts optimization model

If we want to know which parts are to be doubled, the following optimization model can be used. First we introduce the presumption of the model.

Let us suppose n parts of the system (aggregates, components) $Z_1, Z_2, \ldots, Z_n$. Each of these parts is characterized by:

p_i and $\overline{p}_i$ − probabilities of the failure-free run of Z_i without and a with reserve,

q_i and $\overline{q}_i$ − the mean value of losses caused by Z_i's disorders without and with a reserve,

[1]University of Economics Prague, Departement of Econometrics, nám W. Churchilla 4, 130 67, Praha 3, Czech Republic, pelikan@vse.cz
[2]University of Economics Prague, Departement of Mathematics, Ekonomická 957, 148 00, Praha 4, Czech Republic, henzler@vse.cz

c_i − costs of the purchase and maintenance of the reserve for Z_i.

Next we suppose:

- a statistical independence of the failures of those parts,
- the costs of the parts' doubling are limited to the amount K.

Let us introduce 0-1 variables $x_1, x_2, \ldots, x_n$, the variable x_i involves the decision between the doubling of Z_i ($x_i = 1$) or not ($x_i = 0$), see [3]. Total costs of the reserves for the parts are $\sum_{i=1}^{n} c_i x_i$ and since the resources for reserves are limited by K, so it has to be valid

$$\sum_{i=1}^{n} c_i x_i \leq K. \tag{1}$$

On that conditions we can:

a) maximize the reliability of the system, that is failure-free run,

b) minimize the mean value of the sum of losses caused by the part's disorders.

In the case a) the probability of the failure-free state of the system is the product of the probabilities of the failure-free states of all the parts.

The part Z_i will be failure-free with the probability $\overline{p}_i$, if it has a reserve ($x_i = 1$). If the part Z_i is without reserve ($x_i = 0$) then the failure-free probability is p_i. Altogether the probability of the part Z_i's failure-free state can be put in the form $p_i + (\overline{p}_i - p_i)x_i$ (see [1]). Hence the total probability of the failure-free state expresses the whole system's reliability:

$$r(x) = \prod_{i=1}^{n} [p_i + (\overline{p}_i - p_i)x_i]. \tag{2}$$

After logarithming the function $r(x)$ in order to make the objective function linear we get the objective function in the form

$$\log r(x) = \sum_{i=1}^{n} \log[p_i + (\overline{p}_i - p_i)x_i]. \tag{3}$$

Since the expression $\log[p_i + (\overline{p}_i - p_i)x_i]$ for $x_i = 0$ equals $\log(p_i)$ and for $x_i = 1$ equals $\log(\overline{p}_i)$, we can write the expression $\log[p_i + (\overline{p}_i - p_i)x_i]$ in the form $(1 - x_i)\log(p_i) + x_i \log(\overline{p}_i)$":

$$\log r(x) = \sum_{i=1}^{n} [(1 - x_i)\log(p_i) + x_i \log(\overline{p}_i)] = \sum_{i=1}^{n} \log(p_i) + \sum_{i=1}^{n} x_i \log(\overline{p}_i / p_i) \tag{4}$$

The maximum reliability model is as follows:

$$\log r(x) = \sum_{i=1}^{n} [\log(p_i) + x_i \log(\overline{p}_i / p_i)], \qquad \sum_{i=1}^{n} c_i x_i \leq K, \quad x_i \in \{0, 1\}, \quad i = 1, 2, ..., n. \tag{5}$$

In the case b) the mean value of the losses caused by the part Z_i's failure is without the reserve q_i and with the reserve $\overline{q}_i$. The mean value of the total losses is as follows:

$$z(x) = \sum_{i=1}^{n} [(1 - x_i)q_i + x_i \overline{q}_i] = \sum_{i=1}^{n} [q_i - x_i \Delta q_i], \qquad \text{where } \Delta q_i = q_i - \overline{q}_i. \tag{6}$$

The minimal losses model is as follows:

$$z(x) = \sum_{i=1}^{n} [q_i - x_i \Delta q_i] \longrightarrow \min, \qquad \sum_{i=1}^{n} c_i x_i \leq K, \quad x_i \in \{0, 1\}, \quad i = 1, 2, ..., n. \tag{7}$$

3 Two models

Now, two models can be distinguished. The first one is a one-case situation, when the function operates for a short period - static case. The second one is a long-time model for a longer time period - dynamic model.

3.1 Static model

The probability of the components' failure Z_i will be denoted as π_i. Consequently the probability of the failure-free run of the part without a reserve is $p_i = 1 - \pi_i$ and the probability of the failure-free run of the part with a reserve is $\overline{p}_i = 1 - \pi_i^2$.

Let the loss caused by one failure of the part Z_i be denoted by Q_i; then the mean value of the loss is

$q_i = \pi_i Q_i$ in case when there is no reserve,

$\overline{q_i} = \pi_i^2 Q_i$ in case when there is a reserve for the part Z_i.

Example 1. Let us have parts Z_1, Z_2, Z_3, Z_4, Z_5. Main characteristics of those parts are contained in Table 1. The costs of the parts' doubling are limited by the amount $K = 100$.

	Z_1	Z_2	Z_3	Z_4	Z_5
p_i	0.9	0.8	0.9	0.93	0.91
$\pi_i = 1 - p_i$	0.1	0.2	0.1	0.07	0.09
π_i^2	0.01	0.04	0.01	0.0049	0.0081
$\overline{p}_i = 1 - \pi_i^2$	0.99	0.96	0.99	0.9951	0.9919
c_i	80	30	35	50	20
Q_i	1666	250	333	1613	989
$q_i = Q_i \pi_i$	166	50	33	113	89
$\overline{q}_i = Q_i \pi_i^2$	16.6	10	3.3	8	8
$\Delta q_i = q_i - \overline{q}_i$	150	40	30	105	81

Table 1 Characteristics of the static model

Reliability model which maximizes the failure-free probability is:

$$\log(r(x)) = \log(0.548402) + x_1 \log(0.99/0.9) + x_2 \log(0.96/0.8) + \\ + x_3 \log(0.99/0.9) + x_4 \log(0.9951/0.93) + x_5 \log(0.91/0.9919) \longrightarrow \max, \quad (8)$$

$$80x_1 + 30x_2 + 35x_3 + 50x_4 + 20x_5 \leq 100, \qquad x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}. \quad (9)$$

By using standard software LINGO we get the optimal solution $x = (0, 1, 1, 0, 1)$ with the failure-free probability equal 0.789041, which is maximal. From the result follows that it has to double Z_2, Z_3, Z_5.

Model which minimizes the mean value of the total losses is:

$$z(x) = 451.9 - 150x_1 - 40x_2 - 30x_3 - 105x_4 - 81x_5 \longrightarrow \min, \quad (10)$$

$$80x_1 + 30x_2 + 35x_3 + 50x_4 + 20x_5 \leq 100, \qquad x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}. \quad (11)$$

When we use again LINGO system, we get the optimal solution $x = (1, 0, 0, 0, 1)$ with the minimal value of losses 220.9 in the mean value. According to this solution only the parts Z_1 and Z_5 will be doubled. The differences in the solutions obtained above we can explain by great influence of the amount of the loss in the optimal solution in the second model. First solution $x = (0, 1, 1, 0, 1)$ means the most reliable system, but the loss is not minimal. Second solution $x = (1, 0, 0, 0, 1)$ gives us less reliable system, but the loss is minimal.

We can observe the values of reliability, mean loss and costs of reserves for different solutions in the Table 2. The values with a bullet are optimal for $K = 100$.

solution x	reliability	losses	doubling cost
(0,0,0,0,0)	0.578402	451.9	0
(0,1,1,0,1)	• 0.789041	300.3	85
(1,0,0,0,1)	0.657534	• 220.6	100
(1,1,1,1,1)	0.934507	45.9	215

Table 2 Static model - different solutions

3.2 Dynamic model

Let us suppose that the system's reliability should be optimized within a period $< 0, T >$ and the periods between the failures of the parts are exponentially distributed. Let the mean value of the period between two failures of the component Z_i be $1/\lambda_i$. The period of replacing the failed component Z_i by a new one has the fixed length t_i (see Figure 1).

If the Z_i is without a reserve, then the probability of the failure-free run due the part Z_i is

$$p_i = \exp(-\lambda_i T). \tag{12}$$

As the number of the failures within the period $< 0, T >$ is Poisson distributed with the mean value $\lambda_i T$, the mean value of the loss caused by the part Z_i in the case $x_i = 0$ is equal to

$$q_i = \lambda_i T Q_i, \text{ where the } Q_i \text{ is the loss caused by one failure of the part } Z_i. \tag{13}$$

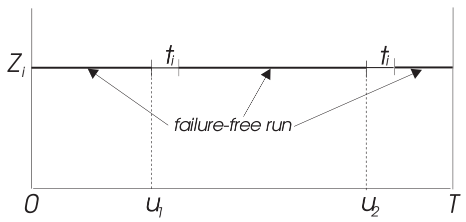

Figure 1 Two failures, u_1, u_2, in the system without a reserve part

In case that component Z_i has a reserve ($x_i = 1$), the t_i is period necessary for the reserve components' exchange, where $t_i << T$ (see Figure 2). During the period t_i the failure-free probability of the system (i.e. no failure of the reserve part) is $\exp(-\lambda_i t_i)$, see [2].

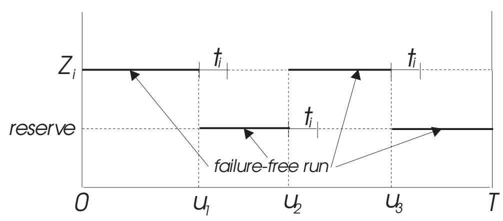

Figure 2 Failures in system with a reserve part

As the failures of a part are treated as Poisson events and therefore independent, the probability of failure-free run of the system in case of k failures of the part Z_i or its reserve is

$$(\exp(-\lambda_i t_i))^k = \exp(-k\lambda_i t_i). \tag{14}$$

Therefore, the overall probability of failure-free run of the system consisting of the part Z_i with a reserve in the period $< 0, T >$ is as follows

$$\overline{p}_i = \exp(-\lambda_i T) \sum_{k=0}^{\left[\frac{T}{t_i}\right]} \frac{(\lambda_i T)^k}{k!} \exp(-k\lambda_i t_i) = \exp(-\lambda_i T) \sum_{k=0}^{\left[\frac{T}{t_i}\right]} \frac{(\lambda_i T \cdot \exp(-\lambda_i t_i))^k}{k!} \approx \exp(\lambda_i T[\exp(-\lambda_i t_i)-1]), \tag{15}$$

where

$\left[\frac{T}{t_i}\right], \quad t_i << T$	is the top number of failures in the period $< 0, T >$,
$\exp(-\lambda_i T)\frac{(\lambda_i T)^k}{k!}$	is the probability of k failures of the part Z_i or its reserve,

Let us find the logarithm of the reliability function $r(x)$; using the general formula (5) and supposing $t_i << T$ we have

$$\log r(x) = \sum_{i=1}^{n} \log(p_i) + \sum_{i=1}^{n} x_i \log(\overline{p}_i/p_i) = \sum_{i=1}^{n}(-\lambda_i T) + \tag{16}$$

$$+\sum_{i=1}^{n} x_i \log \sum_{k=0}^{\left[\frac{T}{t_i}\right]} \frac{(\lambda_i T)^k}{k!} \exp(-k\lambda_i t_i) \approx -T\sum_{i=1}^{n}\lambda_i + T\sum_{i=1}^{n} x_i\lambda_i \exp(-\lambda_i t_i).$$

The mean value of the number of system failures due to the component Z_i's, i.e. $\overline{q_i}$ can be established in the following way.

The number of failures both in the part Z_i and its reserve is Poisson distributed. The time between the two failures is exponentially distributed with parameter λ_i. The system fails if the time interval between the failures of the part Z_i and its reserve (or vice versa) is shorter then the repair interval t_i. The probability of this event is equal $1 - \exp(-\lambda_i t_i)$. Let the number of failures of the part Z_i in the interval $< 0, T >$ be k. These failures can be taken as binomial events, with one outcome " failure of the reserve of Z_i within interval t_i, i.e. system failure" and other outcome "no such failure within t_i. Then probability $P(X_i = r)$, where X_i is the number of system failures due to Z_i and its reserve, is as follows:

$$P(X_i = r) = \sum_{k=r}^{\left[\frac{T}{t_i}\right]} \binom{k}{r} (1-\exp(-\lambda_i t_i))^r (\exp(-\lambda_i t_i))^{k-r} \cdot \frac{(\lambda_i T)^k}{k!} \exp(-\lambda_i T) =$$

$$= (1-\exp(-\lambda_i t_i))^r \frac{(\lambda_i T)^r}{r!} \exp(-\lambda_i T) \sum_{k=r}^{\left[\frac{T}{t_i}\right]} \frac{(\lambda_i T)^{k-r}}{(k-r)!} \exp(-(k-r)\lambda_i t_i), \tag{17}$$

$$E(X_i) = \sum_{r=0}^{\left[\frac{T}{t_i}\right]} r \cdot P(X_i = r) = (1-\exp(-\lambda_i t_i))\,\lambda_i T \exp(-\lambda_i T) \sum_{r=1}^{\left[\frac{T}{t_i}\right]} (1-\exp(-\lambda_i t_i))^{r-1} \frac{(\lambda_i T)^{r-1}}{(r-1)!} \cdot$$

$$\cdot \sum_{k=r}^{\left[\frac{T}{t_i}\right]} \frac{(\lambda_i T)^{k-r}}{(k-r)!} \exp(-(k-r)\lambda_i t_i) = (1-\exp(-\lambda_i t_i))\,\lambda_i T \exp(-\lambda_i T) \sum_{s=0}^{\left[\frac{T}{t_i}\right]-1} \frac{(\lambda_i T)^s}{s!} \approx (1-\exp(-\lambda_i t_i))\,\lambda_i T,$$

$$\overline{q}_i = E(X_i) \cdot Q_i = (1-\exp(-\lambda_i t_i))\,\lambda_i T Q_i. \tag{18}$$

Example 2. Let us have parts Z_1, Z_2, Z_3, Z_4, Z_5 (see Table 3 for characteristics; values of c_i and Q_i are the same as in the static example). Let the overall time be $T = 5$ [in hours]. The costs of the parts' doubling are limited by the amount $K = 100$.

Reliability model which maximizes the failure-free probability is:

$$\log(r(x)) = \sum_{i=1}^{n} \log(p_i) + \sum_{i=1}^{n} x_i \log(\overline{p}_i/p_i)] \approx -T\sum_{i=1}^{n}\lambda_i + T\sum_{i=1}^{n} x_i\lambda_i \exp(-\lambda_i t_i) = \tag{19}$$

$$= -8.5 + x_1 \cdot 1.921579 + x_2 \cdot 1.446960 + +x_3 \cdot 1.701680 + x_4 \cdot 2.160696 + x_5 \cdot 0.978240 \longrightarrow \max,$$

	Z_1	Z_2	Z_3	Z_4	Z_5
λ_i	0.4	0.3	0.35	0.45	0.2
t_i [in hours]	0.1	0.12	0.08	0.09	0.11
p_i	0.135335	0.223130	0.173774	0.105399	0.367879
$\overline{p}_i$	0.924574	0.948342	0.952828	0.914567	0.978475
$\log(\overline{p}_i/p_i)$	1.921579	1.446960	1.701680	2.160696	0.978240
c_i	80	30	35	50	20
Q_i	1666	250	333	1613	989
q_i	3332	375	582.75	3629.25	989
$\overline{q}_i$	130.649	13.2599	16.0907	144.048	21.5204
$\Delta q_i = q_i - \overline{q}_i$	3201.35	361.740	566.659	3485.20	967.480

Table 3 Characteristics of the dynamic model

$$80x_1 + 30x_2 + 35x_3 + 50x_4 + 20x_5 \leq 100, \qquad x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}. \tag{20}$$

By using LINGO software we get the optimal solution $x = (0, 1, 0, 1, 1)$ with the failure-free probability equal 0,019958, which is maximal. From the result follows that it has to double Z_2, Z_4, Z_5.

Model which minimizes the mean value of the total losses is:

$$z(x) = 8908 - 3201.35x_1 - 361.740x_2 - 566.659x_3 - 3485.20x_4 - 967.480x_5 \longrightarrow \min, \tag{21}$$

$$80x_1 + 30x_2 + 35x_3 + 50x_4 + 20x_5 \leq 100, \qquad x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}. \tag{22}$$

By LINGO we get we get the optimal solution $x = (0, 1, 0, 1, 1)$ with the minimum loss equal 4093.580.

In contrary to the static model, in the dynamic model the optimal solution on respect to both maximum reliability and minimum loss is the same (see Table 4 for different solutions).

solution x	reliability	losses	doubling cost
(0,0,0,0,0)	0.000203	8908	0
(0,1,0,1,1)	0.019958	4093.580	100
(1,1,1,1,1)	0.252368	325.571	215

Table 4 Dynamic model - different solutions

Acknowledgements

This research was supported by GAČR grant No. GACR P403/12/1947 and the project F4/18/2011 founded by the Internal Grant Agency of the University of Economics, Prague.

References

[1] Černý, M.: Binary Segmention and Bonferroni-Type Bounds, *Kybernetika* **47** (2011), No. 1, 38–49. ISSN 0023-5954.

[2] Dupač, V.: Private communication.

[3] Nemhauser, G. L., Wolsey, L. A.: *Integer and Combinatorical Optimization.* John Wiley & Sons Inc., New York, 1988.