Max-plus algebra at road transportation

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Abstract. In this paper two models of two transportation problems over the max-plus algebra are analysed. First problem deals with coordination of strongly dependent light crossroads. Second problem focuses on computing bus line timetables and on synchronization of departures from some interchange stops at bus transportation network. The behaviour of this discrete-event dynamic systems can be modeled as linear systems in max-plus algebra with operations max and plus. We show how it is possible to use eigenvalues and eigenvectors of matrix for computing practical characteristics of these road transport systems. Some results of computation experiments using open source software ScicosLab with real data of Czech town Protějov instances of problems are presented.

Keywords: max-plus algebra, eigenproblems, discrete-event dynamic systems, light crossroads, bus line timetables

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1 Introduction

Max-plus algebra is an attractive way to describe a class of non-linear problems that appear for instance in discrete event dynamic systems [11, 5]. In this paper two models of two transportation problems over max-plus algebra are analysed. First problem deals with coordination of strongly dependent light crossroads [7]. Second problem focuses on computing bus line timetables and on synchronization of departures from some interchange stops at bus transportation network [9]. Similar approach based on eigenproblem can be found in [3]. We show how it is possible to use eigenvalues and eigenvectors of matrix for optimizing practical characteristics of these road transport systems. Every computation is done via open software ScicosLab [6]. We begin with some known facts from the theory of the max-plus algebra.

2 Max-plus algebra

We will use needed notation and formulation from [1, 4]. We suppose that the reader is not familiar with the basic definition for max-plus algebra. An extensive discution of the max-plus algebra can be found in [4].

Let $\Re$ be the set of real numbers, $\varepsilon = -\infty, e = 0$, $\Re_{\max} = \Re \cup \{\varepsilon\}$, $\underline{n} = \{1, 2, \ldots, n\}$ where $n \geq 2$ is a natural number.

Let $a, b \in \Re_{\max}$ and define operations $\oplus$ and $\otimes$ by: $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. Let $\Re_{\max}^{n\times n}$ be set of $n \times n$ matrices with coeficients in $\Re_{\max}$. Matrices can be added and multiplied formally in the same manner as in the classical algebra. The sum of matrices $\mathbf{A}, \mathbf{B} \in \Re_{\max}^{n\times n}$ denoted by $\mathbf{A} \oplus \mathbf{B}$ is defined by $(\mathbf{A} \oplus \mathbf{B})_{ij} = a_{ij} \oplus b_{ij} = \max\{a_{ij}, b_{ij}\}$ for $i, j \in \underline{n}$. The product of matrices $\mathbf{A} \in \Re_{\max}^{n\times l}, \mathbf{B} \in \Re_{\max}^{l\times n}$ denoted by $\mathbf{A} \otimes \mathbf{B}$ is defined by $(\mathbf{A} \otimes \mathbf{B})_{ij} = \bigoplus_{k=1}^{l} a_{ik} \oplus b_{kj}$ for $i, j \in \underline{n}$.

The communication graph of matrix plays in our models the fundamental role.

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Let $\mathbf{A} \in \Re_{\max}^{n \times n}$ be any matrix. A weighted digraph $\mathcal{G}(\mathbf{A}) = \big(\mathcal{N}(\mathbf{A}, \mathcal{D}(\mathbf{A})\big)$, where $\mathcal{N}(\mathbf{A}) = \underline{n}$ is set of nodes, $\mathcal{D}(\mathbf{A}) = \{(i,j) \in \underline{n} \times \underline{n} : a_{ji} \neq \varepsilon\}$ is set of arcs and a weight $w(i,j) = a_{ji}$ is associated with any arc is called **communication graph** of $\mathbf{A}$.

A path from node i to node j is a sequence of arcs $p = \{(i_k, j_k) \in \mathcal{D}(\mathbf{A})\}_{k \in \underline{m}}$ such that $i = i_1, j_k = i_{k+1}$ for $k < m$ and $j_m = j$. The path p consists of the nodes $i = i_1, i_2 \ldots, i_m, j_m = j$ with length m denoted by $|p|_1 = m$. In the case when $i = j$ the path is said to be circuit. A circuit is said to be cycle if nodes i_k and i_l are different for $k \neq l$.

Let us denote by $P(i,j;m)$ the set of all paths from node i to node j of length $m \geq 1$ and for any arcs $(i,j) \in \mathcal{D}(\mathbf{A})$ let its weight be given by a_{ji}. Then weight of path $p \in P(i,j;m)$ denoted by $|p|_w$ is defined to be the sum of the weights of all the arcs that belong to the path. The average weight of path p is given by $|p|_w / |p|_1$.

Let $\mathbf{A} \in \Re_{\max}^{n \times n}$ be a matrix. If $\mu \in \Re_{\max}$ is a scalar and $\mathbf{v} \in \Re_{\max}^{n}$ is a vector of at least one finite element such that: $\mathbf{A} \otimes \mathbf{v} = \mu \otimes \mathbf{v}$ then, μ is called an **eigenvalue** and $\mathbf{v}$ an **eigenvector**.

Let $C(\mathbf{A})$ denote the set of all cycles in $\mathcal{G}(\mathbf{A})$ and write: $\lambda(\mathbf{A}) = \max_{p \in C(\mathbf{A})} \frac{|p|_w}{|p|_1}$ for the maximal everage cycle weight. An efficient way of evolution $\lambda(\mathbf{A})$ is Karp's algorithm of complexity $O(n^3)$ or numericaly almost linear Howard's algorithm. A cycle $p \in \mathcal{G}(\mathbf{A})$ is said to be **critical** if its average weight is maximal.

Theorem 1 (Bacceli at al. [1])**.** *Let $\mathbf{A} \in \Re_{\max}^{n \times n}$ is irreducible or equivalent if $\mathcal{D}(\mathbf{A})$ is strongly conected. Then one and only one finite eigenvalue (with possible several eigenvectors) exists. This eigenvalue is equal to the maximal average weight of cycles in $\mathcal{G}(\mathbf{A})$.*

Cuninghame-Green (1960) showed (cited in [2]) that $\lambda(\mathbf{A})$ is an optimal solution of the linear program

$$\begin{aligned} \lambda &\rightarrow \min && && (1)\\ \lambda + x_i - x_j &\geq a_{ij} && \forall (i,j) \in \mathcal{D}(\mathbf{A}), && (2)\\ x_i &\geq 0 && \forall i \in \mathcal{N}(\mathbf{A}). && (3) \end{aligned}$$

Because $\lambda(\mathbf{A})$ is the optimal value of linear program (1-3), following fact is proven.

Theorem 2 (Cechlárová [2])**.** *Let $\mathbf{A} \in \Re_{\max}^{n \times n}$ be a matrix. Then inequality*

$$\mathbf{A} \otimes \mathbf{x} \leq \mu \otimes \mathbf{x} \tag{4}$$

is solvable if and only if $\mu \geq \lambda(\mathbf{A})$.

Following max-plus models use two practical interpretation of the eigenvalue and eigenvector of matrix.

- If the road transport system is performed in cycles and consists of jobs with matrix desribing duration of operations $\mathbf{A}$ that were started according to some eigenvector $\mathbf{x}$ of this matrix, then it will move forward in regular steps. The time elapsed between the consecutive starts of all jobs will be equal to $\lambda(\mathbf{A})$.
- Let us have a schedule for the road transport system that requires time interval between two consecutive jobs smaller than certain value μ. In this case idle times of system can be used to optimize some characteristic of transport elements.

  Let $\mathbf{A} \in \Re_{\max}^{n \times n}, \mu \in \Re$ from Theorem 2 and $f : \Re_{\max}^{n} \rightarrow \Re$ be a real max-plus function. Then we woud like to solve following optimization problem:

$$\begin{aligned} f(\mathbf{x}) &\rightarrow \min && (5)\\ \mathbf{A} \otimes \mathbf{x} &\leq \mu \otimes \mathbf{x}. && (6) \end{aligned}$$

Remark 1. We don't know any publications in this area for max-plus algebra solving problems like (5-6). First computational experiments show that the critical cycle of the communication graph of $\mathbf{A}$ can play fundamental role.

3 Coordination of strongly dependent light crossroads

Managing traffic flow in congested networks requires a coordination of light crossroads and so controlling vehicular and walkers behaviour. In this section we present a basic max-plus model for the coordination of q dependent light crossroads where the constant output transport flow from one model is the input transport flow to the other and vice versa.

We focus on following formulation of problem: *Let's have scheme for strongly dependent light crossroads with traffic flows of vehicles and flows of walkers on crosswalks. Let a phase schemas for first and second crossroad be given. Our goal is to find a traffic light policy for crossings - signal schedule with minimum length of cycle and coordened relations between chosen vehicle flows.*

Inputs for the following mathematical model are:

- q – number of crossroads,
- n – number of transport flows,
- m – number of phases of the signal schedule,
- $\mathcal{P}$ – set of transport flows (vehicles and walkers),
- P_i – i^{th} transport flow, $P_i \in \mathcal{P}, i \in \underline{n}$,
- $\mathcal{F}$ – set of phases of the signal schedule,
- F_j^k – j^{th} phase of the signal schedule for k^{th} crossroad, $F_j \in \mathcal{F}, j \in \underline{m}$,
- $m(P_i P_j)$ – mean time between lock-up of P_i flow and lead-off P_j flow,
- $t(P_i)$ – minimum green time for flow P_i,
- τ – minimum green time for coordened flow.

Variables for this model are:

- λ_S – length of the cycle of the signal schedule,
- $x_i(k)$ – begining time of green for the flow P_i in k^{th} phase of the signal schedule.

Let $S_1 = \{(P_i, P_j) \in \mathcal{P} \times \mathcal{P} : P_i \in F_r^c, P_j \in F_{(r \mod n)+1}^c, c \in \underline{2}, r \in \underline{q}\}$ be the set of pair of flows between adjacent phases and let subset S_2 of the coordinated pair of flows be defined similarly. Now we can define matrix $\mathbf{A} \in \Re_{\max}^{n\times n}$ where finite elements a_{ij} give the time between lead-off transport flow P_j and lock-up transport flow P_i:

$$a_{ij} = \begin{cases} m(P_j, P_i) + t(P_j) & \text{if } (P_j, P_i) \in S_1 \\ m(P_j, P_i) + \tau & \text{if } (P_j, P_i) \in S_2 \\ \varepsilon & \text{otherwise} \end{cases} \tag{7}$$

Details can be found in [8] where author studies three types of coordened vehicle flows for two crossings scheme with two crossroads.

Solving equation of the spectral problem $\mathbf{A} \otimes \mathbf{v} = \mu \otimes \mathbf{v}$ we can set $\lambda_S = \mu$ and $\mathbf{x}(0) = \mathbf{v}$. Then the development of the system described above can be expressed by a vector equation of the form

$$\mathbf{x}(k+1) = \mathbf{A} \otimes \mathbf{x}(k).$$

When we set $\mu > \lambda(\mathbf{A})$ then we can use difference $\mu - \lambda(\mathbf{A})$ of coordined flows of crossroads as removind difference between starting green as much as possible while solving optimization problem:

$$\sum_{(P_i,P_j)\in S_2} (x_j - x_i)^2 \quad \to \quad \min \tag{8}$$

$$\mathbf{A} \otimes \mathbf{x} \quad \leq \quad \mu \otimes \mathbf{x}. \tag{9}$$

Remark 2. In general, for solving these problems conditions described in **Remark 1** also apply. In case of searching for one non-negative solution it is possible to formulate this as a problem of quadratic programming.

The earliest computational experiments were done on real data of two dependent crossroads with 12 vehicles flows and 4 walker flows in this way: First each of feasible scheme of phases for single crossroads is generated and each of feasible matrices $\mathbf{A}$ is examined. After that corresponding eigenproblems are solved. Eigenvectors are compared by objective function (8). Via perturbation scheme based on penalize of finite entry a_{ij} defined as $a_{ij} + \delta$ with randomly penalization value δ defined as $0 < \delta \leq \mu - \lambda(\mathbf{A})$ relevant eigenproblem is solved. The best of these solutions is an approximate solution of the problem (8–9). Next computational experiments were done for 3 real types of coordened flows by presented heuristic method. Experiments show that proposed difference $\mu - \lambda(\mathbf{A})$ for our instances can be used so that $f(\mathbf{x})$ object function of (8) can sometimes decrement even to the value $\dfrac{\lambda(\mathbf{A})f(\mathbf{v})}{\mu}$, where $\mathbf{v}$ denotes the best found eigenvector.

4 Synchronization of departures at bus line timetables

This model focuses on computing bus line timetables and on sychronisation of departures from some crossing stops at transportation network. *Let bus network with the set of bus lines, the sets of stops and crossing stops of bus lines be given. We know traveling time on line directions and turnaround time of the lines. We suppose that every line is served by given number of buses and buses do not change lines. The goal is to find the synchronized departure time of timetables when passenger changes lines on crossing stops. The objective function, the maximal difference between departure times of the same directions, is minimized.*

Inputs for the following mathematical model are:

- S – set of stops of bus network,
- S_c – set of crossing stops of bus network; $\emptyset \neq S_c \subset S$,
- q – number of lines in the bus network,
- n – number of line directions in the bus network,
- i – index of line direction; $i \in \underline{n}$,
- α_i – first stop of the line direction i; $\alpha_i \in S$,
- β_i – last stop of the line direction i; $\beta_i \in S$,
- t_i – traveling time on direction i from α_i to β_i; $i \in \underline{n}$,
- L_q – set of directions for q^{th} line; $q \in \underline{q}$,
- o_q – turnaround time of the q^{th} line,
- m_q – number of buses on the q^{th} line,

and variables are:

- $x_i(k)$ – k^{th} bus synchronized departure time in direction i of timetable,
- λ_D – length of the mean period between departure times of timetable.

Now we can define matrix $\mathbf{A} \in \Re_{\max}^{n\times n}$ where finite elements a_{ij} give the time between departure time from stop a_j and direction j and arrival time to stop β_i of direction i.

$$a_{ij} = \begin{cases} \min\{t_j, o_q/m_q\} & \text{if } \beta_i = \alpha_j, i \in L_q, j \in L_q \\ t_j/m_q & \text{if } \beta_i = \alpha_j, i \in L_q, j \notin L_q \\ o_q & \text{if } i = j, i \in L_q, \alpha_i \in S - S_c \\ \varepsilon & \text{otherwise} \end{cases} \tag{10}$$

More details can reader find in Turek's disertation work [10]. Because vector $\mathbf{x}(k) = \big(x_1(k), x_2(k), \dots, x_n(k)\big)$ denotes the time in which all buses started for the k^{th} cycle then if all buses wait for all preceding jobs to finish their operations, the earliest possible starting time at $(k+1)^{th}$ cycle are expressed by vector $\mathbf{x}(k+1)$, which can be expressed by vector equation ower max-plus algebra in the form

$$\mathbf{x}(k+1) = \mathbf{A} \otimes \mathbf{x}(k). \tag{11}$$

If an initial condition

$$\mathbf{x}(0) = \mathbf{x}, \tag{12}$$

is given, the whole future evolution of (11) is determined. When $\mathbf{x}$ is an eigenvector and λ_D is an eigenvalue of the matrix $\mathbf{A}$ then in general

$$\mathbf{x}(k+1) = \mathbf{A}^k \otimes \mathbf{x}(0) = \lambda_D^k \otimes \mathbf{x}. \tag{13}$$

So the mean waiting time at crossing stops of unit intensity of passengers flow has in our system the value $w(\mathbf{A})$, which

$$w(\mathbf{A}) = \frac{\sum_{i\in\underline{n}} \sum_{j\in\underline{n}:a_{ij}>\varepsilon:\beta_j=\alpha_i\in S_c} \big(\lambda_D + x_i - x_j - a_{ij}\big)}{\sum_{i\in\underline{n}} \sum_{j\in\underline{n}:a_{ij}>\varepsilon:\beta_j=\alpha_i\in S_c} 1}. \tag{14}$$

It is possible to show that matrix $\mathbf{A}$ defined by (10) is irreducible and also from Theorem 2 only one λ_D exists that can be computed as maximal average cycle weight $\lambda(\mathbf{A})$. If eigenvector associated with λ_D is fractional number, then it is inappropiate for any real timetable. In this case we can solve

$$\mathbf{A} \otimes \mathbf{y} = \mu \otimes \mathbf{y}, \tag{15}$$

where $\mathbf{y}$ is approximate integer eigenvector for given approximate eigenvalue μ. Corresponding approximate mean waiting time of crossing stops can be calculated analogically to the form (14).

Note when we set $\mu > \lambda_D$ then we can use difference $\mu - \lambda_D$ to remove differences betweenwaiting time for transfering passengers while solving this optimization problem:

$$\sum_{i\in\underline{n}} \sum_{j\in\underline{n}:a_{ij}>\varepsilon:\beta_j=\alpha_i\in S_c} (\mu + x_i - x_j - a_{ij})^2 \quad \rightarrow \quad \min \tag{16}$$

$$\mathbf{A} \otimes \mathbf{x} \quad \leq \quad \mu \otimes \mathbf{x} \tag{17}$$

$$\mathbf{x} \quad \text{is} \quad \text{integer.} \tag{18}$$

Remark 3. For solving these problems same conditions as described in **Remark 2** apply with additional integer requirement for solution. The question is if simplier method for solving this than method of mixed quadratic programming exists.

The computational experiments are motivated by a problem of real removing one of three bus lines with one crossing stop. We begin with construction of matrix $\mathbf{A}$ for simple instance with 2 lines where every line is served by one bus. After solving corresponding eigenproblems and solutions we can create timetable for every stop of lines with these eigenvectors. There are different ways of constructing matrix $\mathbf{A}$. If we suppose first stops of buses is predetermined then length of mean period between departure times of timetable can be different. For matrix $\mathbf{A}$ defined by (10) we obtain minimal values $\lambda(\mathbf{A})$ and $w(\mathbf{A})$ but none of eigenvectors is integer. This is interesting for timetables in different peaks and troughs of day with distinct periods. This fact is not surprising for more buses on lines. The only difference against previous problem (5-6) is that solution $\mathbf{x}$ of relevant problem must be integer. Experiment done with 3 lines with 2 and 3 crossing stops by this heuristic method shows that difference $\mu - \lambda_D)$ can be again used to minimize object function (16). Feasible integer solutions were found only in 83 instances of 100.

5 Conclusion

This paper introduced two models for solving the practical problems from road transport based on spectral theory of max-plus algebra. The first computational experiments of Prostějov instances [8, 10] for both max-plus models show that this approach can be applied and generates open max-plus optimization problems.

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