Effects of heavy tails on optimal investment and consumption

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Abstract. We study the effect of heavy tails on optimal proportion and consumption problem, i.e. we compare the optimal and Merton proportion and consumption and compute the wealth loss. We state and show that the effect of heavy tails is quite slight in usual conditions. The effect stays nonsignificant even if we contaminate the Lévy measure of the risky asset dynamics by severe drops of price. However, we observe that heavy tails need to be taken into account if an investor is exposed to a very huge loss or even bancruptcy. This could be the case of a very aggressive investor. Finally, we study the lower bound of the optimal investment proportion. We show that even for infinite kurtosis the optimal investment proportion is still positive. We also studied the rate of convergence to zero of the optimal investment proportion as the volatility/risk averse coefficient approaches infinity or expected return approaches zero.

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1 Introduction

Optimal consumption and allocation of wealth between a risky asset and a bank account is a crucial question for any investor. This problem dates back to Merton [6], who models the risky asset by geometric Brownian motion. However, empirical tests reveals that the distribution of relative asset returns may be far from gaussian. They are characteristic by heavy tails and asymmetry. If we allow jumps in the model, we obtain heavy tailed with possibly asymmetric distribution. Note also that it is natural not to assume that the trajectory of asset price is continuous because the information which influence the price reveals discontinously.

The theoretical formulas for the optimal investment and consumption strategy for a jump-diffusion process and under CRRA utility function were derived in [5]. Oksendal in [8] in example 3.1 shows that adding jumps decreases the optimal proportion. The effect on optimal consumption differs for aggressive (with risk averse coefficient lower than 1) and conservative investors. The aggressive would consume smaller proportion relative to Merton while the conservative bigger. Cvitanić [4] performs a numerical study and suggests a Variance Gamma process as a reasonable model for risky asset price dynamics. A different model for the risky asset price dynamics, Normal Inverse gaussian process, is used in [9], where the optimal consumption proportion is also empirically studied. Benth et al. in [2] calibrates Normal Inverse Gaussian to the financial data and then applies the optimal control for different values of risk free rate.

The numerical study in [9] indicates that the effect of jumps is very slight for usual values of skewness and kurtosis. Furthermore, the effect is still quite insignificant for extremely high kurtosis and becomes significant for a very negatively skewed asset returns. In the following paper we show that the effect of heavy tails increases rapidly as the investor approaches 'the bankruptcy region'. We also prove that even for infinite kurtosis the optimal proportion is still positive. Furthermore, we study limits of the optimal relative to Merton proportion as moments and risk averse coefficient approach boundary values.

The paper is organized as follows. In the section 2 the economic model and its solution are presented.

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Section 3 deals with numerical application. The results of paper Petrásek [9] are briefly described and extended by a model with additional source of risk. Finally, section 4 is devoted to the study of lower bound of the investment proportion and its consequences are discussed.

2 Theoretical Model

Consider an investor placing his money into two assets, risk-free, paying interest rate r, and an risky asset that follows a geometric Lévy process

$$\mathrm{d}\,S_t = S(t-)\left(\alpha\,\mathrm{d}\,t + \sigma\,\mathrm{d}\,W_t + \int_{-1}^{\infty} z\widetilde{N}(\mathrm{d}\,t,\mathrm{d}\,z)\right), \quad t > 0, \tag{1}$$

where W is Brownian motion and $\widetilde{N}$ is process of compensated Poisson random measure, both on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$.

At any time $t \geq 0$ an investor can choose the number Δ_t of risky assets S_t in his portfolio and he can also consume money from his account at rate $C_t \geq 0$. Let further $\theta_t = \frac{\Delta_t S_{t-}}{X_{t-}}$ denotes the proportion of his capital invested in risky asset at time t and $c_t = \frac{C_t}{X_{t-}}$ the consumption proportion. Then the value of his portfolio X_t is given by the stochastic differential equation

$$\begin{aligned}
\mathrm{d}\,X_t &= \Delta_t\,\mathrm{d}\,S_t + rX_t\,\mathrm{d}\,t - C_t\,\mathrm{d}\,t && (2)\\
&= X_{t-}\theta_t\left(\alpha\,\mathrm{d}\,t + \sigma\,\mathrm{d}\,W_t + \int_{-1}^{\infty} z\widetilde{N}(\mathrm{d}\,t,\mathrm{d}\,z)\right) + rX_t\,\mathrm{d}\,t - c_tX_t\,\mathrm{d}\,t, && (3)
\end{aligned}$$

with $X(0) = x$, $\theta_t, c_t \in \mathcal{F}_{t-}$ (predictable).

Definition 1. An $\mathcal{F}_{t-}$ adapted policy (Δ_t, C_t) is admissible if $C_t \geq 0$ and X_t, $t \geq 0$ is the wealth and consumption portfolio (3). We denote $\mathcal{A}(x)$ a set of admissible policies with $X_0 = x$.

The objective of an investor is to maximize his discounted consumption C_t, i.e. we seek a function v (called a *value function*) such that

$$v(x) = \sup_{(\theta_t,c_t)\in\mathcal{A}(x)} \int_0^{\infty} e^{-\beta t}\,\mathrm{E}\,U(C_t)\,\mathrm{d}\,t, \tag{4}$$

where β is a discount factor and U denotes a power utility function of the form

$$\begin{aligned}
U(x) &= \frac{x^{1-p}}{1-p}, \quad p > 0, p \neq 1,\\
&= \log(x), \quad p = 1.
\end{aligned}$$

Theorem 1 (Optimal Consumption and Portfolio)**.** *Assume the portfolio (3) and the objective (4) and assume that the risk aversion coefficient* $p \neq 1$. *Let us denote*

$$\Lambda(\theta) = \alpha_t\theta(1-p) - \frac{1}{2}\sigma_t^2\theta^2 p(1-p) + \int_{-1}^{\infty}\left((1+\theta z)^{1-p} - 1 - \theta z(1-p)\right)\nu_t(dz)$$

and let θ^* *be such that*

$$\Lambda'(\theta^*) = 0. \tag{5}$$

Assume also that

$$\beta - r(1-p) - \Lambda(\theta^*) > 0.$$

Then θ^* *is the optimal investment proportion,* $c^* = (K(1-p))^{-1/p}$ *is the optimal consumption and* $v(z) = Kz^{1-p}$ *is the value function, where* $K = \frac{1}{1-p}\left(\frac{\beta - r(1-p) - \Lambda(\theta^*)}{p}\right)^{-p}$.

Proof. The proof is based on using the dynamic programming principle and solving the acquired Hamilton-Jacobi-Bellman equation. For a general theorem (Optimal Control for Jump Diffusions) see [8], theorem 3.1 and subsequent example 3.2 or [2]. □

The theorem solves the investors problem, however, leads to an integro-differential equation, which must be solved numerically in general.

3 Empirical results

The empirical study performed in [9] shows quite slight effect of heavy tails. The numerical study proceeded as follows. The process of logarithmic prices L_t of the risky asset was modeled by a Normal Inverse Gaussian process, see [1], i.e. it can be described by a pure jump Lévy process of the form

$$\mathrm{d}\,L_t = b_L\,\mathrm{d}\,t + \int_{\mathbb{R}} z\widetilde{N}_L(\mathrm{d}\,t, \mathrm{d}\,z), \quad 0 \le t < \infty,$$

where $\widetilde{N}_L$ is a compensated Poisson random measure with Lévy measure ν_L. Applying Itô formula we get the dynamics of the prices (1)

$$\mathrm{d}\,S_t = \mathrm{d}\exp(L_t) = S_{t^-}\left\{\left(b_L + \int_{\mathbb{R}} (e^z - 1 - z)\,\nu_L(z)\right)\mathrm{d}\,t + \int_{\mathbb{R}} (e^z - 1)\,\widetilde{N}_L(\mathrm{d}\,t, \mathrm{d}\,z)\right\}.$$

Finally, the equation (5) was solved.

The numerical study aimed at the effect of skewness and kurtosis on the optimal proportion relative to the Merton proportion θ_0. Let us fix the risk aversion coefficient $p = 6$, volatility $\sigma = 16$ % (p.a.), Sharpe ratio of the risky asset as 0.40, risk free rate 2 % (p.a.) and discount parameter β 10 %. Then the optimal proportion relative to the Merton proportion was more than 99.5 % for symmetric returns with a very high excess kurtosis (30). The effect of skewness was much stronger with decrease of the optimal proportion of almost 2 % for skewness level equal to -1. The results are summed up in table 1.

We extend the study by computing a *wealth loss* w. Wealth loss is the initial added wealth the investor ignoring jumps in the model requires such that he has the same utility from discounted consumption as if he allocates and consumes his money optimally, i.e. we solve the following equation[1]

$$v\left(x(1+w); \theta_0, c_0\right) = v(x; \theta^*, c^*),$$

where $v(x, \theta, c)$ is the value function for a given strategy (θ, c) with initial wealth x. We can see that an investor ignoring jumps requires only additional \$1.4 per \$10000 for negatively skewed relative returns and excess kurtosis 30.

σ	κ_3	κ_4	θ^*/θ_0	c^*/c_0	w
0.160	-1.000	30.000	98.096	99.700	0.000143
0.160	-0.000	30.000	99.510	99.939	0.000008
0.160	1.000	30.000	101.008	100.188	0.000040
0.160	-1.500	60.000	96.943	99.522	0.000381
0.160	-0.000	60.000	99.028	99.877	0.000034
0.160	1.500	60.001	101.312	100.256	0.000069

Table 1: Optimal consumption and investment proportion for different values of skewness and kurtosis. κ_3 denotes skewness and κ_4 kurtosis of asset returns.

An investor must also face to an unobservable effects. These are the events that appears with very low frequency, and so are left out from the model[2]. We conclude that our model lacks severe drops of asset price. If we assume, that such events can be described by an independent Lévy process, we can use the fact that if we have two independent Lévy processes they never jump at the same time (corollary of Theorem 4.1 and Theorem 5.1 in [3]). In other words if we have two independent processes $\{L_t\}$ and $\{E_t\}$ with Lévy measures ν_L and ν_Y then the Lévy measure of their sum is defined as

$$\nu_X(B) = \nu_L(B) + \nu_E(B), \quad \forall B \in \mathbb{R}\,.$$

[1] Another possibility how to compare optimal and suboptimal strategy is to compute the certainty equivalence for both strategies.

[2] Generally, models assume stationary time series. Thus they are calibrated using quite a short (stationary) time period.

We assume that the process of unobservable events $\{E_t\}$ can be modeled by a Poisson compound process with a very low intensity and gaussian jumps[3], i.e.

$$E_t = \sum_{k=0}^{N_t} Y_k - \lambda \mu_E t, \quad Y_k \sim N(\mu_E, \sigma_E^2),\ N_t \sim Po(\lambda t).$$

Parameters are set so that the intensity of such events is every four years and the day relative loss is approximately 10 %[4]. The Lévy measure of this process is given by the formula

$$\nu_E(x) = \lambda \cdot \frac{1}{\sqrt{2\pi}\sigma_E} \cdot \exp\left(-\frac{(x-\mu_E)^2}{2\sigma_E^2}\right), \quad x \in \mathbb{R}\,. \tag{6}$$

We performed a similar numerical study as previously, but with additional source of risk, given by the process of unobservable events $\{E_t\}$. We found that the effect of additional jumps is almost negligible, see table 2. The contamination reduces the investment proportion into the risky asset by about 0.5 % and the wealth loss is still only about \$2.4 per \$10000 invested for negatively skewed relative returns. We can conclude that the effects of higher moments are not so serious in usual conditions which is consistent with Cvitanić [4]. Cvitanić [4] also observed, that increasing volatility with Merton proportion kept fixed increases the difference between optimal and Merton investment proportion. Note that, it is equivalent with increasing the Sharpe ratio, whose strong effect was observed in [9]. In the following section we will study the effect of risk aversion coefficient.

σ	κ_3	κ_4	θ^*/θ_0	c^*/c_0	w
0.160	-1.227	33.219	97.582	99.612	0.000236
0.160	-0.227	33.219	99.007	99.856	0.000037
0.160	0.773	33.219	100.474	100.102	0.000010
0.160	-1.727	63.219	96.482	99.446	0.000511
0.160	-0.227	63.219	98.546	99.797	0.000080
0.160	1.273	63.220	99.814	99.929	0.000003

Table 2: Optimal consumption and investment proportion for different values of skewness and kurtosis. κ_3 denotes skewness and κ_4 kurtosis of asset returns.

4 Poisson type models

We will use compound Poisson models possibly with diffusion part. Note that as long as jumps are allowed in the model, the investment proportion cannot exceed the level $1/J$, where J is the biggest possible drop of price, i.e. if the model allows 50% loss of the risky asset price our investment proportion cannot exceed 2. On the other hand, jumps (for moderate values of volatility, skewness and kurtosis) reduce the optimal investment proportion only by a several percent.

Let us assume the following model for asset price dynamics

$$\mathrm{d}\, S_t = S_{t-}\left(\alpha\, \mathrm{d}\, t + \sigma_C\, \mathrm{d}\, W_t + J\, \mathrm{d}\, \widetilde{N}_t\right), \tag{7}$$

where J is the given jump size and $\widetilde{N}_t$ is a compensated Poisson process with intensity λ. Thus the volatility of relative returns is $\sigma = \sigma_C + \lambda \cdot J^2$. We set $\lambda = 0.0001$ (per day), jump -0.3, then skewness equals -2.605 and excess kurtosis equals 77.248.

In picture 1 we can see that the effect of added jump is almost negligible for a conservative investor with risk aversion coefficient higher than 3 and becomes extremely significant as the risk aversion decreases below the value 1. Jumps significantly influence the optimal proportion as well as the optimal consumption as the investor approaches the border of solvency region. The optimal relative to Merton consumption is higher for aggressive investor with $p < 1$.

[3]Poisson compound process with gaussian jumps plus diffusion part is called Merton model, see [7].

[4]$\lambda = 0.001$ (per day) $\mu_E - 0.1$ (per day), $\sigma_E = 0.05$ (per day).

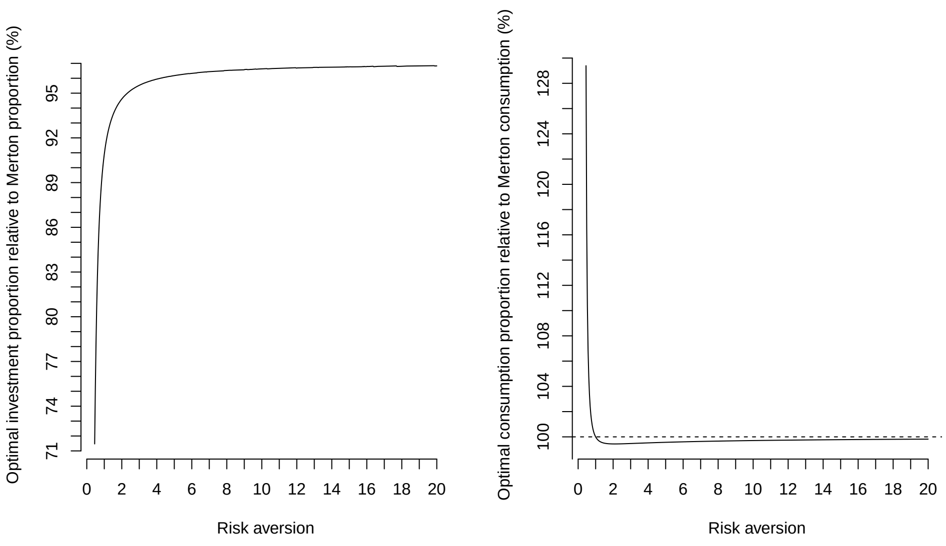

Figure 1: Optimal proportion for model (7) relative to Merton proportion and optimal consumption proportion for model (7)relative to Merton proportion consumption as a function of risk aversion coefficient.

We can ask whether there exist a distribution of risky asset returns that leads to zero optimal proportion. It is enough to think about the worst scenario for the investor, where the risky asset follows a compound Poisson process with a Dirac measure at 100% loss. The Lévy measure of the described model is given by the formula

$$\nu(z) = \lambda \cdot \delta_{-1}(z),$$

where δ_x denotes a Dirac measure at point x and λ, which determines variance σ^2, is set such that the volatility is still 16 % (p.a.). If we plug the proposed Lévy measure into the equation (5) we see that the optimal proportion solves the equation

$$\alpha - \sigma^2 \cdot \left(\frac{1}{(1-\theta)^p} - 1 \right) \quad = \quad 0.$$

Proposition 2. *The lower bound for the investment proportion is*

$$\theta_{LB} = 1 - \left(1 + \frac{\alpha - r}{\sigma^2} \right)^{-1/p}.$$

Note that we found the optimal proportion lower bound for any risky asset model of the type (1) (geometric Lévy process), thus also for models with infinite kurtosis.

Proposition 3. *Assume a model, where the only risk source is* 100% *loss. Then*

$$\begin{aligned} \frac{\theta_{LB}}{\theta_0} &\to 1 \quad as \quad \frac{\alpha - r}{\sigma^2} \to 0, \\ \frac{\theta_{LB}}{\theta_0} &\to \frac{\sigma^2}{\alpha - r} \cdot \log\left(\frac{\alpha - r}{\sigma^2} + 1 \right) \quad as \quad p \to \infty, \end{aligned}$$

for all other parameters being fixed.

Proof. Taylor expansion in $\frac{\alpha - r}{\sigma^2}$ of the first order is

$$\theta_{LB} = \frac{\alpha - r}{p\sigma^2} + o\left(\frac{\alpha - r}{\sigma^2} \right)$$

and in $1/p$, we obtain

$$\theta_{LB} = \frac{1}{p} \log\left(\frac{\alpha - r}{\sigma^2} + 1 \right) + o\left(\frac{1}{p} \right).$$

□

We conclude that if a risky asset follows a geometric Lévy process. Then

$$\frac{\theta^*}{\theta_0} \to \gamma \quad \text{as} \quad \frac{\alpha - r}{\sigma^2} \to 0,$$

where $\gamma \geq 1$ and the rate of convergence (to zero) of θ^* and θ_0 as $p \to \infty$ differs.

5 Conclusion

In this paper we studied the effect of heavy tails on optimal investment proportion and consumption. We continued in the study by Petrásek [9]. We extended the results by adding a severe drops of price to the original risky asset price dynamics. We conclude that the influence of heavy tails is still very slight. However, it may be significant for an aggressive investor or if the risky asset has a very high Sharpe ratio. We observed that as an aggressive investor reaches the borders of solvency region the optimal relative to Merton investment proportion decreases very rapidly while the optimal relative to Merton consumption proportion increases. Furthermore, we included the *wealth loss* in the numerical study and inferred that a suboptimal investor (using Merton proportion and consumption under heavy tailed and possibly negatively skewed risky asset returns) loses only a few money units per 10 000 units invested.

Finally, we derived the lower bound for the investment proportion. We proved that even for infinite kurtosis the optimal proportion is still positive. Furthermore, we proved that as the excess expected return of the risky asset approaches zero or its volatility approaches infinity the optimal proportion does converge to zero at the same rate as Merton investment proportion. Thus Merton investment proportion can serve as a good approximation in such cases. But as the risk aversion coefficient goes to infinity the rate of convergence generally differs. Hence if an investor holds a very small proportion in the risky asset due to his high risk aversion, he cannot neglect jumps in general.

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