Usage of the extremal algebra in solving the travelling salesman problem

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Abstract. This article compares many ways of solving the traveling salesman problem. At first classical heuristic methods and methods using graph theory are mentioned. At the first part many universal methods are described, which can be also used in other transportation problems. The traveling salesman problem is solved by using genetic algorithm in the second part of this article. This algorithm generates at the beginning the first generation, chooses five thousand parents by the roulette method, crosses these pairs and determines the next generation. This process continues with next generations until stabilization. The algorithm is demonstrated on two examples. At the final part extremal algebras, max-plus algebra and max-min algebra, are defined and illustrated by examples. Monge matrices and some their properties are described in these algebras. The optimization of the traveling salesman problem, which leads to a reduction of the computation complexity, is described for these matrices. The optimization is solved at first for the classical case and then for the matrices that satisfy the strict Monge property. Matrices with the strict Monge property lead to the linear complexity of the problem.

Keywords: Monge matrix, max-plus algebra, max-min algebra, travelling salesman problem, genetic algorithm.

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1 Introduction

Different ways of solving the travelling salesman problem are compared in this paper. Classical heuristic algorithms, genetic algorithms and usage of Monge matrix in extremal algebras are described.

2 The travelling salesman problem

The travelling salesman problem can be in practice encountered very often. In comparison with realization each travel separately, the ring connection can save a great deal of costs in cases where exists the necessity to distribute a certain material from one or more producers to many consumers.

The travelling salesman problem represents the problem of finding the shortest ring connection among M cities where their distance is known in advance. Each distance in fact represents the cost of every route (the assumption is that the costs to travel from city A to city B are the same as vice versa). The goal is to minimalize the total costs.

The problem doesn't lie in discovering the algorithm of finding the shortest connection – the easiest approach is obvious: to search among all the circuits among the given cities and choose the shortest of them.

The problem lies in the fact that with the mounting number of cities the number of potential circuits increases very quickly. Thus the time needed to compute the aforesaid brutal force solution becomes quite unacceptable already with a couple of dozen cities.

Resources to this chapter are [1], [5], [6], [9]. The next chapter contains the listing of known methods to solve this problem.

2.1 Standard solving methods

The travelling salesman problem belongs to NP-hard tasks. These tasks can be easily formulated but they are

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difficult to solve. They can be solved using heuristic methods, general methods, methods based on the graph theory or by means of genetic algorithms.

2.2 Heuristic methods

The outcome of heuristic methods can be considered to be an acceptable solution, but the value of the objective function may not be optimal. This fact is their principal disadvantage. Contrary to exact methods, the heuristic methods are fast, polynomial and it is easy to use them to solve even the complex tasks. The main and essential disadvantage is the fact that they don't guarantee discovering the solution which is the global optimum. Another important disadvantage is the impossibility to determine the estimation of the error.

Heuristic methods can be split to two methods. First are the solutions composing methods which construct the solution from the beginning. The other ones are the solution improving methods which are based on a certain solution, which is then improved by iterative procedures. Solution composing methods can be further split to methods of the sequential procedures, which operate locally and take into consideration only the vicinity of the partial result and methods of the parallel procedures which operate globally – they start at several places simultaneously and then combine the partial solutions into the global one.

2.3 General methods

These methods can be commonly used even for different types of transport tasks provided that the costs of the travel from city A to city B are the same as the costs from city B to city A.

The disadvantage of these methods is the fact that there is not known any estimation of precision of the solution.

2.4 Methods based on the graph theory

The second big group of algorithms which is possible to use for solving the traveling salesman problem are methods based on the graph theory. Fastest are greedy algorithms which work very simple. At first the edge with minimal costs is chosen. This edge is added to result way and is removed from further computing.

2.5 Evolution methods

Next way how to solve the travelling salesman problem is usage of the evolution algorithms. It is suitable to use the natural representation. For example the way B - A - D - E – C is represented as a chromosome string (b a d e c). Usage of special selection operators is also advantageous. One of the most suitable is recombination operator with edge crossing - it is the most successful operator for solving the travelling salesman problem. Its specialty is that two parents create only one descendant. Each segment of route comes from one of the parents. For each city the edge table stores information about its neighbors. List of neighbors comes from its parents.

One of the cities with minimum number of neighbors is randomly is chosen. This city is added to result way and is deleted from the edge table and then algorithm repeats this procedure again.

Algorithm finds the minimal possible ring way between specified number of points. Roulette wheel is used for choosing parents. Scales of roulette wheel are inverse proportional to the length of the way. The ways with smaller length have bigger rating on the scales. For example half of the distance has double bigger rating.

Chosen pairs of parents are crossed by selection operators with the edge recombination. This crossing creates one new descendant. Only in the case that the length of its way is smaller than length of the way of his parents then descendant is part of the next generation otherwise parent with minimal length of the way is part of the next generation. This algorithm continues until the certain accuracy is reached. Our example shows that after few steps there is no further improvement of length of the way. This algorithm can lead only to discover of the local optimum instead of the global optimum.

3 Implementation of genetic algorithm

- First generation is randomly generated
- 5000 pairs of parents are chosen by roulette wheel method
- Crossing of parents and forwarding to next generation one of parents or descendent – depends on shortest way
- Repeating this algorithm until stable state is reached. Local optimum – shortest path is found.

This algorithm was programed in the Pearl and in the Matlab. The results of the experiments are shown at following part of this paper.

3.1 Results of the first experiment

10 cities (square net 1000 x 1000), 5000 parents, 5 generations (after that stable state has been reached)

Cities: A 744 299, B 88 294, C 663 996, D 981 654, E 509 554, F 387 92, G 547 410, H 806 168, I 420 186, J 227 864

- Generation 0 - J I F B E G A H D C = 3619.28216907124
- Generation 1 - F I B E J C D A H G = 3567.40119270441
- Generation 2 - J B F I E G A H D C = 3384.84611373796
- Generation 3 - A H I F B G E J C D = 3382.63430361154
- Generation 4 - F B J C D E G A H I = 3358.5860296155
- Generation 5 - F B J C D E G A H I = 3358.5860296155

As you can see in generation 4 the shortest route was found.

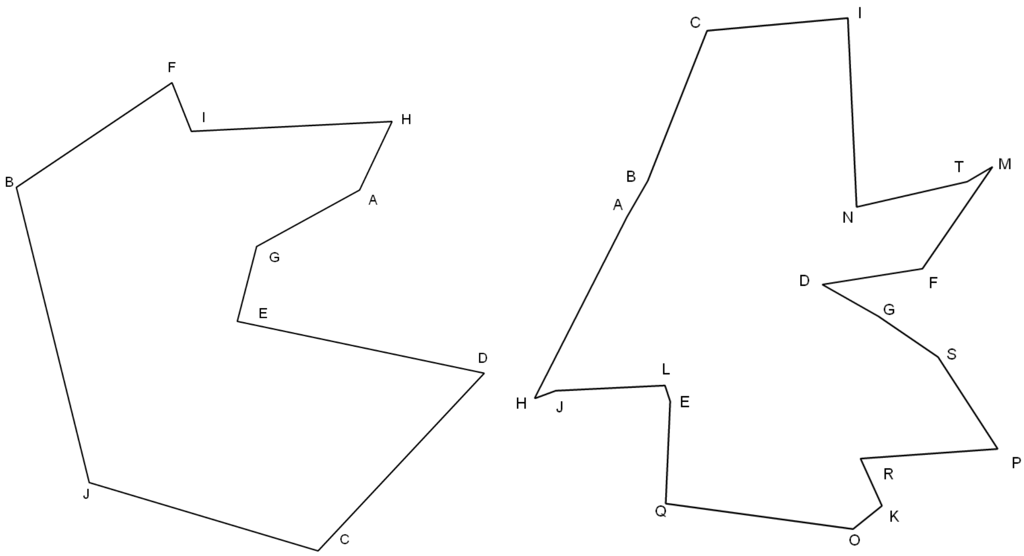
3.2 Results of the second experiment

20 cities (square net 1000 x 1000), 5000 parents, 20 generations (after that stable state has been reached)

Cities: A 200 350, B 237 288, C 340 29, D 537 468, E 275 668, F 709 441, G 635 522, H 42 663, I 583 7, J 77 651, K 639 851, L 266 642, M 832 263, N 595 333, O 590 890, P 841 752, Q 268 845, R 604 768, S 737 592, T 788 290

- Generation 0 - E A B C I Q J L H T R P G S D M F N K O = 6455.4108624135
- Generation 1 - Q P S N T G E B C I M F O K L R D A J H = 6041.12292335198
- Generation 2 - F M Q E L J H T N I C A B P D S G K O R = 5876.67183903036
- Generation 3 - P R S G T M D F O K N I C B A E Q L J H = 5149.89044555545
- Generation 4 - P R S G T M D F O K N I C B A E Q L J H = 5149.89044555545
- Generation 5 - F D R K O S T M P N Q E H J L A B C I G= 5064.88710405238
- Generation 6 - F M T N R G S I C B A J H Q E L K O P D = 4696.05677939314
- Generation 7 - S R P K O Q E J H L A B D G F M T N C I = 4363.16256524023
- Generation 8 - S G T M N F P D I C B A E L J H Q K O R = 4317.18528791843
- Generation 9 - S F T M N D G I C A B L E J H Q K O P R = 4152.42657949083
- Generation 10 - F S P O K D R Q E L J H A B C I N T M G = 4129.020628350
- Generation 11 - Q E L H J A B C I T M N F S P D G K O R = 3923.424919074
- Generation 12 - O K P R S D G F N T M I C B A L E Q J H = 3967.5052191970
- Generation 13 - O K P R S D G F N T M I C B A L E Q J H = 3967.5052191970
- Generation 14 - S G F N D T M I C B A J H E L Q K O P R = 3852.7588661828
- Generation 15 - S G F N D T M I C B A J H E L Q K O P R = 3852.7588661828
- Generation 16 - C B A H J L E Q O K R P S G D F M T N I = 3484.2585837226
- Generation 17 - C B A H J L E Q O K R P S G D F M T N I = 3484.2585837226
- Generation 18 - C B A H J L E Q O K R P S G D F M T N I = 3484.2585837226
- Generation 19 - C B A H J L E Q O K R P S G D F M T N I = 3484.2585837226
- Generation 20 - C B A H J L E Q O K R P S G D F M T N I = 3484.2585837226

The shortest route was found in generation 16.



Picture 1: Illustration of the first and the second experiment – the shortest route

In comparison with solutions using extremal algebras, this algorithm is very fast but it can find sometimes only the local optimum instead of the global optimum. By using the Monge matrices in extremal algebras, used in next chapter, we are always able to find the optimal solution in the polynomial time.

4 Extremal algebras

In the next part of this paper extremal algebras are described. Following text is based on information from [2], [3], [4], [8].

4.1 What is max-plus algebra

Max–plus algebra $(\bar{R}, \oplus, \otimes)$ is an algebraic structure with two binary operations $\oplus$, $\otimes$ and a set $\bar{R} = R \cup \{-\infty, \infty\}$, which is an extension of real numbers. The operations maximum and plus in max-plus algebra are derived from the linear ordering in the set of real numbers. Max-plus algebra belongs to the family of so-called extremal algebras.

The operations in max-min algebra are defined as follows:

$$\text{for } x, y \in \bar{R}:\ x \oplus y = \max(x, y),\ x \otimes y = x + y.$$

For matrices A, B over $\bar{R}$ we can define operations $\oplus$, $\otimes$ analogously as in linear algebra over R with addition and multiplication. We assume matrices A, B of suitable types.

Example 1.

$$(4 \quad 8 \quad 2) \otimes \begin{pmatrix} 5 \\ 3 \\ 9 \end{pmatrix} = (4 \otimes 5) \oplus (8 \otimes 3) \oplus (2 \otimes 9) = 9 \oplus 11 \oplus 11 = 11$$

4.2 What is max-min algebra

Analogically we can define max-min algebra, where operations are defined as follows:

$$\text{for } x, y \in \bar{R}:\ x \oplus y = \max(x, y),\ x \otimes y = \min(x, y).$$

Example 2.

$$(4 \quad 8 \quad 2) \otimes \begin{pmatrix} 5 \\ 3 \\ 9 \end{pmatrix} = (4 \otimes 5) \oplus (8 \otimes 3) \oplus (2 \otimes 9) = 4 \oplus 3 \oplus 2 = 4$$

4.3 Monge matrices in max-min algebra

Monge matrices in max-min algebra are special matrices which hold following condition: for all elements $a \in A$ and $i, j, k, l \in \mathbb{N}$, where A is a matrix of type (m, n), i, k are row indexes, where $i < k$ and j, l are column indexes, where $j < l$:

$$a_{i,j} \otimes a_{k,l} \leq a_{i,l} \otimes a_{j,k}$$

This condition has to be fulfilled for all row indexes i, k and all columns indexes j, l, which means, that this condition has to be fulfilled by all quaternion of such elements from given matrix, which are positioned in this matrix in rectangle.

In max-plus algebra we can define Monge matrices in the similar way by using operations maximum and plus.

4.4 Usage of the Monge matrices for solving the travelling salesman problem

It is shown that it is possible to solve the travelling salesman problem in polynomial time easily. We assume that setting matrix of this problem is Monge. This problem is studied in [3], [2], [8]. Solving for Monge matrices are based on the pyramidal way concept, which is defined as following:

Let assume that way $\phi = \langle 1, i_1, i_2, \ldots, i_r, n, j_1, \ldots, j_{n-r-2} \rangle$ is pyramidal, if following condition holds true:

$$i_1 < i_2 < \cdots < i_r, j_1 > j_2 > \cdots > j_{n-r-2}$$

Then we can formulate following sentence which reduce the computation complexity for this special type of matrices into the quadratic. The problem is studied in [3], where is described the following Theorem.

Theorem 1. The optimal route which is pyramidal exists if the matrix consisting of the costs of the ways between the cities, in max-plus algebra, is Monge. This route can be found by using the dynamic programing with the algorithm of the computation complexity $O(n^2)$.

There can be also other special cases. If distances between all cities have value 1, than computation complexity for solving travelling salesman problem is linear.

4.5 The travelling salesman problem in max-min algebra

The travelling salesman problem in the max-min algebra is the defined analogically as in the max-plus algebra, although the interpretation is different. Graph can be interpreted as a net. The value on each edge represents the size of the minimal flow on this edge. Finding ring connection with maximal throughput is the problem. In max-min algebra analogical statement as Theorem 1**,** can be formulated:

Theorem 2. The optimal route exists if the matrix consisting of the costs of the ways between the cities in max-min algebra is Monge. This route can be found by using the dynamic programing with the algorithm of the computation complexity of $O(n^2)$.

The traveling salesman problem can be solved most effectively for strict Monge matrices. For this special type of matrices the traveling salesman problem can be resolved in the linear time with the algorithm of the computation complexity of O(n). Detailed information, including the Theorem 2, can be found at [3], [8], [10].

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