# Elimination of Regional Economic Disparities as Optimal Control Problem

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**Abstract.** The paper deals with an issue related to the existence of regions and their economic development. Particularly the problem of possible reduction of the existing economic regional disparities is solved. Regional disparities are understood here only as the difference among the economic productivity of the regions. It is assumed that a part of the public income of the more productive regions is shifted into the area of infrastructure of the less productive regions in order to reduce disparities in regional productivity and differences in incomes between populations of regions. The model that is formulated as optimal control problems in continuous time provides necessary conditions for success of this aim of regional politics and it shows how to reduce the difference between productivity of regions. To find optimal conditions Pontryagin maximum principle is used.

**Keywords:** region, regional development, regional disparities, redistribution, allocation of investment, optimal control, Pontryagin maximum principle.

**JEL classification:** R11, C61
**AMS classification:** 49J15

## 1 Introduction

The aim of regional policy is regional development aimed at increasing of their coherence and their competitiveness. Because it is more efficient to allocate investment and concentrate economic activity in those regions that have higher productivity, see e.g. [8], regions that have lower productivity may be lagged behind the more efficient regions. Such regions can then become less competitive. For the sake of maintaining cohesion of regions it would be useful to make some redistribution of income between more productive and less productive regions. In order to diminish possible differences between regions and their productivity, governments aimed its investment to infrastructure to the less developed regions. The aim of such efforts is to strengthen the capital stock of the population of the regions, helping to improve the technological level of the region and thus enhance the competitiveness of such regions, cf. [5]. The paper will present a mathematical model that deals with income redistribution and regions investing in their infrastructure. The model is based on ideas in [3], [6] and [7].

## 2 Problem Description and Model

Consider an economic unit with one central government, which consists of two regions. One of the region is considered to be more developed and its residents will have higher incomes, while the second region is considered to be less developed and its residents will have lower incomes. Quantities describing the richer region will be denoted by index $r$ (rich) and variables related to the poorer region will be denoted by index $p$ (poor). The model is based on the assumptions of free capital and immobile labor, it means that no dramatic migration of workers to work in a richer region is considered.

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### 2.1 Households

Assume that all households in both regions have the same logarithmic utility function and the same discount rate $\rho$, $\rho \in (0,1)$. This means that the utility functional of representative households in the region $i$, $i \in \{r,p\}$, is defined as

$$U_i = \int_0^\infty e^{-\rho t} \cdot \ln c_i(t) dt, \tag{1}$$

where $c_i(t)$ represents household consumption in time $t$, $t \in [0,\infty)$. Assume further that household income in time $t$ is determined by the wage $w_i(t)$ and by inscribed interest from the assets $a_i(t)$ of the household, it means that for the household budget constraint can be written the following equation

$$\dot{a}_i(t) = r_i(t)a_i(t) + w_i(t) - c_i(t), \tag{2}$$

where $r_i(t)$, $r_i(t) \in (0,1)$, is the interest rate in the region $i$, $i \in \{r,p\}$, at time $t$, $t \in [0,\infty)$. Households seek to maximize the utility functional (1) subject to the condition (2). This problem is an optimal control problem with state variable $a(\cdot)$ and control variable $c(\cdot)$ that can be solved by Pontryagin maximum principle, see [4]. The Hamiltonian for this problem is

$$H(a_i, c_i, p_i) = e^{-\rho t} \ln c_i + p_i(r_i + w_i - c_i),$$

where we omit variable $t$ denoting time and $p_i$, $p_i = p_i(t)$, is the adjoint function of the optimal control problem. Observing that $H$ is a concave function in the variable $c_i$ and that it is differentiable we can apply the first-order condition to get

$$\frac{\partial H}{\partial c_i} = e^{-\rho t} \cdot \frac{1}{c_i} - p_i = 0.$$

The adjoint equation is

$$\dot{p}_i = -\frac{\partial H}{\partial a_i} = -p_i a_i.$$

If the result from the first relation is substituted to the adjoint equation it yields

$$\frac{\dot{c}_i(t)}{c_i(t)} = r_i(t) - \rho, \tag{3}$$

where $i \in \{r,p\}$. For the given problem the following transverzality condition can be considered

$$\lim_{t\to\infty} \frac{a_i(t)}{c_i(t)} e^{-\rho t} = 0. \tag{4}$$

Linear ordinary differential equation (2) can be solved by integrating factor method. Employing this method and relations (3) and (4) we finally gain

$$c_i(t) = \rho \cdot (a_i(t) - h_i(t)), \tag{5}$$

where $i \in \{r,p\}$ and $h_i(t)$ is the household wealth defined as

$$h_i(t) = \int_t^\infty w_i(s) \exp\left(-\int_t^s r_i(v) dv\right) ds. \tag{6}$$

### 2.2 Firms

Consider that in each region $i$, $i \in \{r,p\}$, there are a large number of competing firms that use private capital $K_i$ and labor $L_i$ for their production $Y_i$. The aggregate product is then used for consumption and private and public investment. We assume that firms have the same Cobb-Douglas production function

$$Y_i = A_i K_i^\alpha L_i^{1-\alpha}, \tag{7}$$

where $A_i$ represents a general exogenous level of technology in region $i$ that can be partly determined by productive government spending, see [3]. Exponents $\alpha$ and $1-\alpha$ respectively represents the coefficient of elasticity due to the production of capital and due to the production of labor respectively.

In relation to the considerations undertaken it can be assumed that the technological level of production depends on the infrastructure of the region. This dependence, which significantly affects the form of the model, will be specified later. For the time evolution of private capital the standard relationship is assumed

$$\dot{K}_i(t) = I_i(t) - \delta K_i(t), \tag{8}$$

where $I_i$ represents gross investment in private capital in region $i$, $i \in \{r, p\}$, at the instant $t$, $t \in [0, \infty)$, and $\delta$ represents the rate of capital depreciation. If we consider installation costs of investment and normalize the price per unit of capital the net flow of present value of the company's profit in each region $i$, $i \in \{r, p\}$, can be expressed as the following functional

$$V_i = \int_0^\infty [(1-\tau)\{A_i(t)K_i(t)^\alpha L_i(t)^{1-\alpha} - w_i(t)L_i(t)\} - I_i(t)] \cdot \exp\left(-\int_0^t r_i(v)dv\right) dt, \tag{9}$$

where $\tau$, $\tau \in (0, 1)$, is the tax rate defined by the central government, $w_i(t)$ is the labor cost per unit of labor in region $i$ at time $t$ and $r_i(t)$ is the interest rate in region $i$ at time $t$. Firm's aim is to maximize (9) subject to the constraint (8). This problem is an optimal control problem in which the control variables are labor $L_i$ and investment $I_i$. Capital $K_i$ is the state variable. Now we apply Potryagin maximum principle, see [4]. The Hamiltonian of the given optimal control problem is

$$H(K_i, L_i, I_i, p_i) = [(1-\tau)\{A_i K_i^\alpha L_i^{1-\alpha} - w_i L_i\} - I_i] \cdot \exp\left(-\int_0^t r_i(v)dv\right) + p_i \cdot (I_i - \delta K_i),$$

where we omit variable $t$ denoting time and $p_i$, $p_i = p_i(t)$, is the adjoint function of the optimal control problem. The maximum principle allows to formulate necessary conditions of the first order for interior optima as follows

$$\frac{\partial H}{\partial I_i} = -\exp\left(-\int_0^t r_i(v)dv\right) + p_i = 0, \tag{10}$$

$$\frac{\partial H}{\partial L_i} = (1-\tau)\{(1-\alpha)A_i K_i^\alpha L_i^{-\alpha} - w_i\} \cdot \exp\left(-\int_0^t r_i(v)dv\right) = 0. \tag{11}$$

The adjoint equation is

$$\dot{p}_i = -\frac{\partial H}{\partial K_i} = -(1-\tau)\alpha A_i K_i^{\alpha-1} L_i^{1-\alpha} \cdot \exp\left(-\int_0^t r_i(v)dv\right) + p_i\delta. \tag{12}$$

Immediately from (11) we gain

$$(1-\alpha)A_i \left(\frac{K_i}{L_i}\right)^\alpha = w_i. \tag{13}$$

If we combine the relations (10) and (12) we gain

$$(1-\tau)\alpha A_i \left(\frac{K_i}{L_i}\right)^{\alpha-1} = r_i + \delta. \tag{14}$$

To interpret these relations we use definition (7) of production functions in both regions. Since

$$\frac{\partial Y_i}{\partial L_i} = A_i(1-\alpha)K_i^\alpha L_i^{-\alpha} = (1-\alpha)A_i \left(\frac{K_i}{L_i}\right)^\alpha = (1-\alpha)\frac{Y_i}{L_i}$$

the condition (13) implies that the unit price of labor is equal to the marginal production of labor. Similarly, since

$$\frac{\partial Y_i}{\partial K_i} = A_i \alpha K_i^{\alpha-1} L_i^{1-\alpha} = \alpha A_i \left(\frac{K_i}{L_i}\right)^{\alpha-1} = \alpha\frac{Y_i}{K_i},$$

the condition (14) implies that the sum of the current interest rate and depreciation of capital is directly proportional to the marginal production of capital. Because between the two considering regions that constitute one economic unit with one central government is a free movement of private capital, the interest rate is the same in both regions, it means that $r_p(t) = r_r(t)$, $t \in [0, \infty)$. The relation (14) then implies that the relation

$$A_r \left(\frac{K_r(t)}{L_r(t)}\right)^{\alpha-1} = A_p \left(\frac{K_p(t)}{L_p(t)}\right)^{\alpha-1} \tag{15}$$

is valid, where $t \in [0, \infty)$.

### 2.3 Government Sector

The government makes effort to redistribute its income in the amount

$$\tau(Y_r(t) + Y_p(t))$$

to increase the competitiveness of the poorer region. Based on this decision more investment in infrastructure will occur in this region than in the richer region. Public investment in non-production area in this model will not be considered. Let us denote $G_i$ the capital of the region $i$, $i \in \{r, p\}$ in the form of infrastructure and further let us denote $u$, $u \in [0, 1]$, the rate of government redistribution. It means that the government will move the extra production of the richer region of the amount $u\tau Y_r(t)$ to a poorer region. If we assume that infrastructure is an immobile capital, we can write a similar differential equation for it as in the case of time evolution of the private capital as follows

$$\dot{G}_r(t) = \tau(1-u)Y_r(t) - \delta G_r(t), \tag{16}$$

and

$$\dot{G}_p(t) = \tau(Y_p(t) + uY_r(t)) - \delta G_p(t), \tag{17}$$

respectively, where $\delta$, $\delta \in (0, 1)$, represents the capital depreciation as has been mentioned earlier.

The last assumption of the model will be a description of how technological progress occurs in the production function (7). In [1] there is the argument that public spending can impact the production in such a way that the production function exhibits constant returns to scale in both the private and the public capital. These considerations are further modified in monograph [2] where it is considered that public expenditures contribute to production in the same extent as labor. In the present model we assume that public expenditures contribute to building infrastructure, that is regarded as public capital, which contributes to the production of the given region. We therefore assume that the infrastructure enriches labor and that it contributes to the production in the same extent as labor, it means that we consider a model of endogenous technological progress enriching labor in the form

$$A_i(t) = AG_i(t)^{1-\alpha}, \tag{18}$$

where $A$, $A > 0$, is a constant that represents the general level of technology common in both regions. This assumption implies that increasing levels of public spending on infrastructure in the region leads to improve the technological level of the region. This is an essential element of the model. We avoid here to consider a more complex relationship to the technological level. Provided (18) the production function (7) can be written in the form

$$Y_i = AK_i^{\alpha}(G_iL_i)^{1-\alpha}. \tag{19}$$

Now the relation (15) can be rewritten and instead of it we gain the following relation

$$\frac{K_r(t)}{G_r(t)L_r(t)} = \frac{K_p(t)}{G_p(t)L_p(t)}. \tag{20}$$

where $t \in [0, \infty)$. This relation will be used in the following analysis.

## 3 Relative Level of Backwardness of Less Productive Region

Under the above given assumptions, the productivity of regions depends on the level of regional infrastructure. This infrastructure is immobile and it progress and adapts slowly in time. To see how the value of productivity in a region changes during time, we introduce a relative measure of productivity $\Omega(t)$ of the poorer region with respect to richer region as the ratio

$$\Omega(t) = \frac{y_p(t)}{y_r(t)} = \frac{Y_p(t)}{L_p(t)} \cdot \left(\frac{Y_r(t)}{L_r(t)}\right)^{-1}. \tag{21}$$

To see how this variable evolve in time and how it is affected by a government redistribution of its income we further introduce relative growth rate of productivity $\gamma_\Omega(t)$ of the poorer region in relation to the richer region

$$\gamma_\Omega(t) = \frac{\dot{\Omega}(t)}{\Omega(t)}. \tag{22}$$

To be able to work with this concept we need to connect it with relations (20) that represents optimal condition for behavior of firms in the economy and (16), (18) respectively, that represents government redistribution of its income. To simplify calculations we introduce the following ratios

$$\kappa(t) = \frac{K_p(t)}{K_r(t)}, \lambda(t) = \frac{L_p(t)}{L_r(t)}, \gamma(t) = \frac{G_p(t)}{G_r(t)}, \tag{23}$$

where $t \in [0, \infty)$. Now (20) can be written as

$$\kappa(t) = \lambda(t) \cdot \gamma(t) \tag{24}$$

and instead of (21) it is possible to write

$$\Omega(t) = \kappa(t)^{\alpha} \cdot \lambda(t)^{-\alpha} \cdot \gamma(t)^{1-\alpha}. \tag{25}$$

To simplify notation we again omit the variable $t$ in further considerations. From (25) we immediately gain

$$\gamma_\Omega = \frac{\dot{\Omega}}{\Omega} = \alpha\frac{\dot{\kappa}}{\kappa} - \alpha\frac{\dot{\lambda}}{\lambda} + (1-\alpha)\frac{\dot{\gamma}}{\gamma} \cdot \tag{26}$$

To simplify this relation we use (24) that allows us to find that

$$\frac{\dot{\kappa}}{\kappa} = \frac{\dot{\lambda}}{\lambda} + \frac{\dot{\gamma}}{\gamma} \cdot \tag{27}$$

If we substitute this result (27) into (26) and if we use the meaning of the variable $\gamma$ given in (23) we find

$$\gamma_\Omega = \frac{\dot{\gamma}}{\gamma} = \frac{\dot{G}_p}{G_p} - \frac{\dot{G}_r}{G_r} \cdot \tag{28}$$

Now we can apply relations (16) and (18). Substituting them into () we gain the relation

$$\gamma_\Omega = \tau\left(\frac{Y_p + uY_r}{G_p}\right) - \tau(1-u)\frac{Y_r}{G_r} \cdot \tag{29}$$

Using (19) for the poor and the rich region we can rewrite this relation in the following way

$$\gamma_\Omega = \tau A\left(\left(\frac{K_p}{G_pL_p}\right)^{\alpha} L_p + u\left(\frac{K_r}{G_rL_r}\right)^{\alpha}\frac{G_rL_r}{G_p} - (1-u)\left(\frac{K_r}{G_rL_r}\right)^{\alpha} L_r\right). \tag{30}$$

If we finally use (19) again we find the resulting relation

$$\gamma_\Omega = \tau A\left(\frac{K_p}{G_pL_p}\right)^{\alpha}\left(L_p - L_r + uL_r\left(\frac{G_r}{G_p} + 1\right)\right). \tag{31}$$

### 3.1 Interpretation

Let us briefly recall that relation (31) was derived for an economic unit consisting of two regions in which the same interest rate exists and the same general level of technology is applied. It means that both regions may borrow funds under the same conditions and have access to the same technologies. Furthermore, the model assumes that the population of poorer region don't massively move to work in the rich region. The following considerations concern only the relationship of productivity in the poorer and richer region and they do not relate to the aggregate growth of output in the given economic unit.

- Let $u = 0$, it means that there is not a redistribution of the total income of the economic unit. The productivity of less developed region can increase if $L_p > L_r$, cf. relation (31). It means that the poorer region can use more labor than the richer region. If $L_p \approx L_r$ then private capital and its using can't contribute to the increasing of productivity of the less developed region.
- Relation (31) implies that $\gamma_\Omega > 0$ if and only if

$$u > \frac{L_r - L_p}{L_r} \cdot \frac{G_p}{G_r + G_p} \cdot \tag{32}$$

  Since the expression on the right hand side of the above relation is less than one there is always such $u$, $u \in [0, 1]$, that (32) is valid. It means that the government can always find a rate of redistribution $u$ of its aggregate income that can increase the productivity of less developed region.

## 4 Conclusion

In the management of economic units that are divided into regions the central government is facing different problems and challenges. On the one hand it pursues economic goals when it invests into the most productive regions, because such investments are most effective. In this way, however, disparities between different regions may rise and less productive regions may lose their competitiveness. On the other hand, the government pursues political goals, which attempts to eliminate differences in economic development of regions to prevent dissatisfactions of the population in less developed regions. Such dissatisfactions could lead to a reduction of cohesion among regions. In order to eliminate this phenomena the central government can increase the competitiveness of less developed regions when it invests part of their income from the more developed regions into the infrastructure of less developed regions. In the present paper an analytical models that seeks to describe such behavior of central government and to show the effectiveness of such decisions were presented. The model was formulated as an optimal control problem and Pontryagin maximum principle was used to find the necessary conditions of optimal solution. It has been shown that under the assumptions of the model the redistribution of government income can increase the productivity of less productive and less developed region. We can conclude that the politics of redistribution is under the given assumptions efficient.

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