Forecasting Financial Time Series

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Abstract. Density forecast is an estimate of the probability distribution of the possible future values of a random variable. In our research we compare two different approaches for one-step ahead forecasting in stock market indices with parameters from normal and two-piece normal (TPN) distribution. We analyze three stock market indices S&P 500 from New York Stock Exchange, FTSE 100 from London Stock Exchange and PX from Prague Stock Exchange with daily returns data. For improving density forecasts, the parameters of two-piece normal distribution for approximating the asymmetry (negative skewness) in standardized residuals were estimated by the maximum likelihood method. Rolling one-step ahead predictions for the last 200 observations of the future returns using samples from stock market indices time series were made using the appropriate ARMA-GARCH models with two different density forecasting distributions (normal and TPN). Using the parameters from two-piece normal, estimated on rolling samples, the ARMA-GARCH-TPN predictions were obtained. We compare these two approaches of forecasting by using chosen tests.

Keywords: financial time series, asymmetry, volatility, two-piece normal distribution

JEL Classification: C58
AMS Classification: 91B84

1 Introduction

Volatility indicates a period of time series connected with high variability or growing variance [1, 2, 3, 7, 8]. This phenomenon plays a great role in the process of modelling and analysing of financial time series. It is the main element of the procedure of quantifying the general risk of financial assets. Modelling and forecasting volatility are generally used in investment decision process for capturing suitable risk of potential investment portfolio and in the analysis of VaR model or option price derivation [3].

A large number of literature in econometrics has focused on evaluating the forecast accuracy of volatility models [4,5]. In our study we compare two different approaches for modelling financial time series and its volatility using stock market indices - S&P 500 from New York Stock Exchange, FTSE 100 from London Stock Exchange and PX from Prague Stock Exchange. During the process asymmetric features in these daily financial times series must be faced. In terms of our approach, asymmetries in the conditional distributions by employing block bootstrap procedure with conditional distribution were captured and a comparison between forecasting from constant and rolling sample were involved as well. These two approaches via one-step-ahead forecasts of future returns using samples from mentioned stock market indices time series were compared.

The paper is structured as follows. In Section 2, we briefly discuss the volatility models. Section 3 introduces the two-piece normal distribution. In Section 4, we apply maximum likelihood method for estimating parameters of GARCH models with symmetric and asymmetric distribution on stock market indices mentioned above. Section 5 concludes.

2 Volatility Models

Volatility measured by the standard deviation or variance of returns often represents the risk measurement concept in the financial market. There are many given models for modelling and forecasting volatility in the financial time series. As the volatility is usually conditional, the presence of conditional heteroscedasticity in the econometrics models may be inferred. In other words, the residuals of linear regression model embody variable variance and, in addition, the variance of residuals is affected by their past values.

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Firstly we define the basic volatility model developed by R. F. Engle, ARCH(p). Model can be written as follows [9]

$$r_t = \Phi_0 + \Phi_1 r_{t-1} + \Phi_2 r_{t-2} + ... + \Phi_s r_{t-s} + \varepsilon_t \qquad (1a)$$

$$\sigma_t^2 = \omega + a_1\varepsilon_{t-1}^2 + a_2\varepsilon_{t-2}^2 + ... + a_p\varepsilon_{t-p}^2 \qquad (1b)$$

$$\varepsilon_t = \sigma_t e_t, \qquad (1c)$$

where $\{\varepsilon_t\}$ is the conditional heteroskedastic process, or it is the variance of the shocks time variable and depends on the past *p* shocks $\varepsilon_{t-1},...,\varepsilon_{t-p}$, the conditional variance σ_t^2 is set on the basis of the information accessible in time *t*-1, e_t is distributed with standard normal distribution *N*(0,1), $\Phi_0, \Phi_1, ..., \Phi_s, \alpha_1, ..., \alpha_p, \omega$ are the parameters of the model, where $\alpha_1, ..., \alpha_p \geq 0,\ \omega > 0$ conditions ensure the positivity of the conditional variance.

In many cases, ARCH model requires an estimation of many parameters *p*, hence GARCH model was developed by T. P. Bollerslev. As we can see in [2], the improvement of the process lies in adding of lagged conditional variance, and therefore the GARCH model accepts the dependency of conditional variance on previous own lags. Instead of (1b), we can formulate for GARCH (p,q)

$$\sigma_t^2 = \omega + \sum_{p=1}^{P}\alpha_p\varepsilon_{t-p}^2 + \sum_{q=1}^{Q}\beta_p\sigma_{t-q}^2, \qquad (2)$$

where *Q* lags of the conditional variance $\sigma_{t-1}^2, \sigma_{t-2}^2, ..., \sigma_{t-Q}^2$ are included. In both models, the kurtosis of shocks ε_t is greater than the kurtosis of normal as is proved in [9].

Similar to [2], we employ *maximum likelihood method* in order to estimate the volatility model.

3 Two-piece normal distribution

According to [10], if a random variable *X* has a two piece normal distribution, $X \sim SN(\mu, \lambda^2, \tau^2)$, then its probability density function is

$$f(x) = \begin{cases} c\exp\left[-\dfrac{1}{2\lambda^2}(x-\mu)^2\right] & if\ x \leq \mu \\ c\exp\left[-\dfrac{1}{2\tau^2\lambda^2}(x-\mu)^2\right] & if\ x > \mu, \end{cases} \qquad (3)$$

where $c = \sqrt{2/\pi}\lambda^{-1}(1+\tau)^{-1}$.

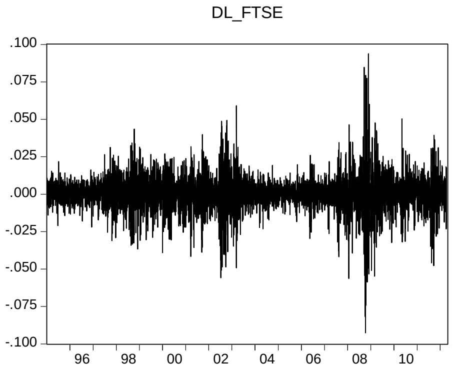
The density of the $SN(\mu, \lambda^2, \tau^2)$ distribution is proportional to the density of the $N(\mu, \lambda^2)$ distribution to the left of the mode μ, to the right of the mode it is proportional to the density of the $N(\mu, \tau^2\lambda^2)$ distribution. This described probability distribution is skewed to the left for $\tau < 1$, to the right for $\tau > 1$ and for $\tau = 1$ it reduces to the usual normal distribution.

4 Application

Three stock market indices S&P 500 from New York Stock Exchange, FTSE 100 from London Stock Exchange and PX from Prague Stock Exchange were chosen for this study. All times series from 3 January 1995 to 27 April 2012 were considered. In order to eliminate the non-stationarity in selected financial times series, we compute the differences between the logarithms of the prices on adjacent days

$$w_t = \ln z_t - \ln z_{t-1} = \ln\frac{z_t}{z_{t-1}}, \qquad (4)$$

where z_t represents the closing price of the particular daily index in period *t*.



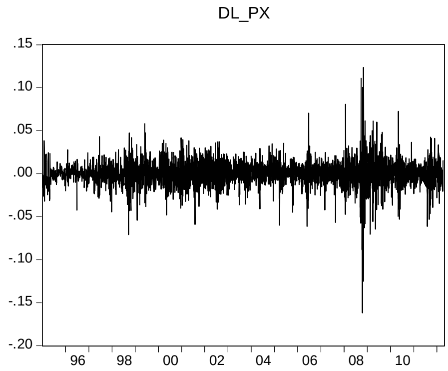


Figure 1 Daily FTSE 100 and PX returns from January 1995 to April 2012

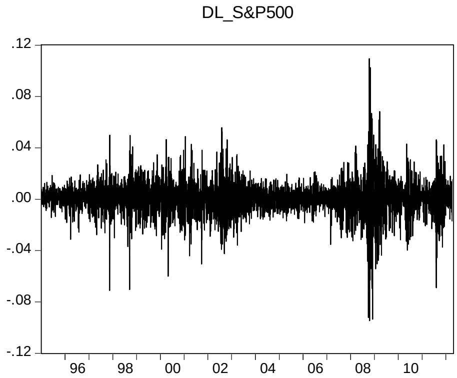


Figure 2 Daily S&P 500 returns from January 1995 to April 2012

Statistic	FTSE 100 returns	PX returns	S&P 500 returns
Observations	4374	4374	4374
Mean	0.00014486	0.000081335	0.00025752
Std. Deviation	0.012249	0.014421	0.012757
Skewness	-0.13847	-0.42767	-0.23466
Excess Kurtosis	5.6562	11.252	7.6101
Minimum	-0.092646	-0.16185	-0.094695
Maximum	0.093842	0.12364	0.10957

Table 1 Descriptive statistics for daily FTSE 100, PX and S&P 500 returns

In the previous graphs (Figure 1 and 2) and table (Table 1), daily returns of each index are slightly negatively skewed and positively biased.

Hua [6] studied two piece normal distribution for obtaining more accurate predictions for symmetric volatility model ARMA-GARCH, where estimated parameters of two-piece normal distribution (TPN) were obtained from residuals received from the mean equation of ARMA-GARCH process. The maximum likelihood method and stationary bootstrap is employed in his study for estimation of TPN. One-step ahead forecast of conditional mean and standard deviations of ARMA-GARCH process with TPN is defined by the formulas below.

Conditional mean of one-step ahead forecast

$$\hat{y}_{t+1} = \hat{\mu}_{t+1} + \hat{\sigma}_{t+1}\hat{\mu}_{\eta}. \tag{5}$$

Conditional standard deviations of one-step ahead forecast

$$\hat{\sigma}_{t+1,1} = \hat{\sigma}_{t+1}\hat{\sigma}_{t+1,\eta 1}, \qquad \hat{\sigma}_{t+1,2} = \hat{\sigma}_{t+1}\hat{\sigma}_{t+1,\eta 2}. \tag{6}$$

Where $\hat{\mu}_{t+1}$ and $\hat{\sigma}_{t+1}$ are the conditional mean and standard deviation of the one-step ahead forecast of ARMA-GARCH model and $\hat{\mu}_{\eta}$ is the mode of two-piece normal distribution, $\hat{\sigma}_{t+1,\eta 1}$ is the standard deviation of the one-step ahead forecast if $x \le \hat{\mu}_{\eta}$ and $\hat{\sigma}_{t+1,\eta 2}$ is the standard deviation of the one-step ahead forecast if $x > \hat{\mu}_{\eta}$. Considering the formula (3) we transform parameters of TPN distribution with this approach $\hat{\sigma}_{t+1,\eta 1} = \lambda$ and $\hat{\sigma}_{t+1,\eta 2} = \tau\lambda$.

In his study [6] Hua concludes that this estimation method provides more accurate estimates for one-step ahead forecast in the financial time series. This proposition is based on the fact that in the case of skewed time series the mode is a better measure of central tendency the mean. In the next section we examine these conclusions.

The returns of indices are stationary under the condition of ADF unit root test [1]. We scrutinise conditional heteroscedasticity by ARCH test and conditional heteroscedasticity of nonlinear type by common SB, PSB, NSP tests [1]. Results from these tests are in Table 2 and Table 3.

Index	**t-statistic**	**Test critical values at 1% level**	**Prob.***
FTSE 100	-30.31201	-3.431663	0.0000
PX	-60.22226	-3.431675	0.0001
S&P500	-50.97834	-3.431666	0.0001

*MacKinnon (1996) one-sided p-values.
Notes: Test critical values -3.43 (1% level), -2.86 (5% level), 2.57 (10% level)

Table 2 ADF unit root tests

Index	**LM test statistic (TR^2)**	**Critical value ($\chi^2_{0.95}(3)$)**
FTSE 100	3183.103	7.8147
S&P 500	3234.073	7.8147
PX	2747.082	7.8147

Notes: Null hypothesis - residuals (conditional mean equation) do not include asymmetry of any type

Table 3 Common SB, PSB, NSB test of asymmetry in conditional heteroscedasticity

According to the results of heteroscedasticity tests, the conditional heteroscedasticity is present in each time series and depends on the size of positive and negative returns, where level of negative shocks affects the conditional heteroscedasticity slightly significantly than the positive shocks do.

We estimate most suitable ARMA-GARCH model for each stock index, for FTSE 100 AR(3)-GARCH(1,1), for S&P 500 ARMA(1,1)-GARCH(2,2) and for PX AR(1),AR(4)-GARCH(1,1), where the constant term is included in each conditional mean equation. The histograms of standardised residuals with skewness -0.495299 (S&P 500), -0.277981 (FTSE 100) and -0.242310 (PX) are in Figure 3.

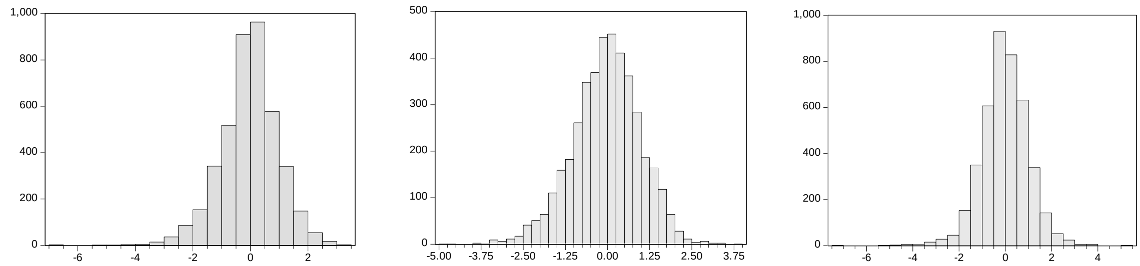


Figure 3 Histograms of standardized residuals for S&P 500, FTSE 100 and PX

Point and interval predictions, based on chosen models, for the last 200 periods (ex post prediction) with rolling sample were constructed. The values of estimated parameters were closed to the parameters gained from the bootstrapped data. All the predictions for ARMA-GARCH-TPN were computed according to formulas (5) - (6).

We employ Q-Q plots and Kolgomorov-Smirnov test for comparison of the quality of ex post prediction between ARMA-GARCH and ARMA-GARCH-TPN. Results from these tests are in the Table 4.

Index: Method	**Kolgomorov-Smirnov test statistic**	**Asymp. *p*-value**
FTSE 100: GARCH	0.4545	1.1570e-018
FTSE 100: GARCH_TPN	0.5354	1.0056e-025
S&P 500: GARCH	0.4141	1.4504e-015
S&P 500: GARCH_TPN	0.5354	1.0056e-025
PX: GARCH	0.3939	4.0040e-014
PX: GARCH_TPN	0.4394	1.8151e-017

Notes: Null hypothesis - empirical and fitted values are prom the same continuous distribution, two-tailed alternative hypothesis.

Table 4 Two-sample Kolgomorov-Smirnov test for rolling sample

Q-Q plot describes the relation between quantiles of the empirical values represented by real-time returns and the point predictions of selected model of each time series. In all cases, the ARMA-GARCH predictions are slightly closer to empirical values. These plots are displayed in the Figure 4 and Figure 5.

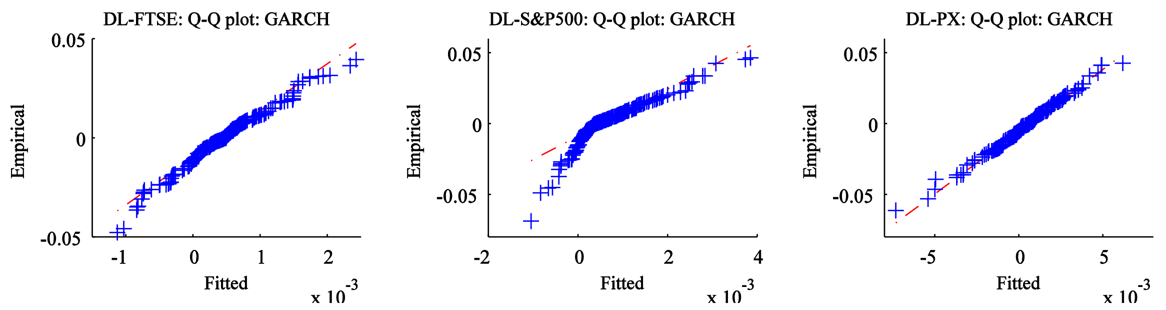


Figure 4 Q-Q plots for one-step ahead forecasts of GARCH with normal distribution for FTSE 100, S&P 500 and PX

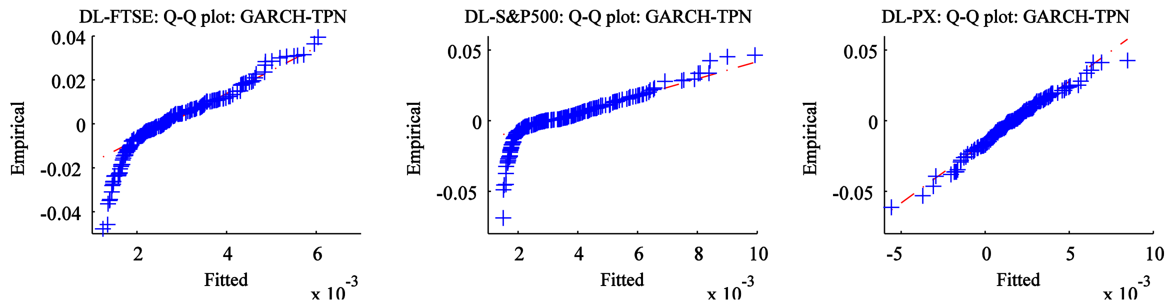


Figure 5 Q-Q plots for one-step ahead forecasts of GARCH with two-piece normal distribution for FTSE 100, S&P 500 and PX

Similar to Hua [6], we applied stationary bootstrap to the standardised residuals from the conditional mean equation and bootstrapped 4 millions data from which the parameters of TPN were estimated by the maximum likelihood estimation and then one-step ahead forecasts for the last 200 periods were conducted. We compared the two approaches of predicting - one-step ahead forecasting from the bootstrapped data sample and appropriate forecasts from the rolling sample. The point and interval predictions were transformed according to formulas (5) - (6). The results from both techniques are only slightly different. Table 5 reports the estimated parameters of TPN distribution obtained from the bootstrapped data.

Index	Mode	Sigma_1	Tau	Sigma_2
FTSE 100	0.169464 (0.001187)	1.12507 (0.000722)	0.768082 (0.001013)	0.86414602
S&P 500	0.193028 (0.001133)	1.147181 (0.000582)	0.72654 (0.000874)	0.83347288
PX	0.075277 (0.000989)	1.066143 (0.000488)	0.869876 (0.000800)	0.92741221

Table 5 Estimated coefficient by MLE for two-piece normal distribution

5 Conclusion

This study deals with modelling and forecasting of three stock indices. Firstly, the condition mean equations were constructed using the linear stationary ARMA processes of Box-Jenkins methodology. Consequently, the appropriate models of symmetric conditional variance (GARCH models) were estimated. For improved density forecasts, the parameters of TPN distribution for approximating the asymmetry (negative skewness) in standardised residuals were estimated by the maximum likelihood method. Rolling one-step-ahead predictions for the last 200 observations were made using the appropriate ARMA-GARCH models. Using the TPN parameters, estimated on rolling samples, the ARMA-GARCH-TPN predictions were obtained. We scrutinise the method developed by Hua [6] and conclusions based on our results are on the contrary to Hua.

Especially in the case of FTSE 100 index, using the TPN transformation, the received predictions fit the empirical values worse than that obtained by using standard ARMA-GARCH models. The main idea of mentioned approach is that the mode should be more relevant and representative measure of central tendency than the mean when studying asymmetric distributions. In our opinion, this methodology could be more convenient in situations, where the skeweness is caused by particular outliers, which is not the case of our samples. The observations are cumulated around the zero value, with several slightly outlying values on both sides (especially during the year 2008 reflecting the financial crisis). Moreover, in finance, the investors usually prefer slightly underestimated yields than underestimated possible losses. However, this method fits more values close to mode, the negative values move further away which could brings huge losses in the case of unexpected negative shocks and it is a drawback in financial analysis.

Acknowledgements

The research project was supported by Grant No. IGA F4/16 /2011, Faculty of Informatics and Statistics, University of Economics, Prague.

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