Measuring transitivity of fuzzy pairwise comparison matrix

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Abstract. A pair-wise comparison matrix is the result of pair-wise comparison a powerful method in multi-criteria optimization. When comparing two elements, the decision maker assigns the value representing the element of the pair-wise comparison matrix. In AHP, the matrix represents a multiplicative preference relation. Here, consistency property plays an essential role. To provide a consistency measure of the pair-wise comparison matrix, the consistency ratio is defined in AHP. In some situations another interpretation is convenient. The preferences can be represented by a fuzzy preference relation, given by the membership function denoting the preference degree (or intensity) of one alternative over the other. The role of consistency is played by the concept of transitivity. In this paper we investigate relations between several types of transitivity of fuzzy relations and multiplicative preference relations. Similarly to the consistency ratio we also define the grade of transitivity. Consequently, we obtain corresponding priority vectors. An illustrative numerical example is supplemented.

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1 Introduction

The pair-wise comparison matrix is a powerful inference tool in decision making (DM), see e.g. [3], [6], that can be also used as a knowledge acquisition technique for knowledge-based systems. It is useful for assessing the relative importance of several objects, when this cannot be done by direct rating. As it is known, most decision processes are based on preference relations, in the sense that they are linked to some degree of preference of any alternative over another. Therefore, to establish properties to be verified by such preference relations is very important for designing good DM models. Three of these properties are investigated in this paper, it is so called reciprocity, consistency property and transitivity property. The lack of consistency or transitivity in DM can lead to wrong conclusions, see [6]. That is why it is important to study conditions under which consistency and/or transitivity is satisfied [3]. On the other hand, perfect consistency/transitivity is difficult to obtain in practice, particularly when evaluating preferences on a set with a large number of alternatives. Then it is important to know, whether and in what grade our preferences are coherent each other. In other words, we ask how strongly are our preferences consistent, or transitive. Hence, our goal here is to derive some simple tools enabling us to measure the grade of consistency/transitivity of pair-wise comparison relations, or, giving us some information about inconsistency of our preferences, i.e. how much consistency/transitivity of our preferences is damaged. As a consequence, we obtain also priority vectors that are used for ranking the variants in DM processes.

2 Multiplicative and additive preferences

The problem can be formulated as follows. Let $X = \{x_1, x_2, ..., x_n\}$ be a finite set of alternatives. These alternatives have to be classified from best to worst, using the information given by a decision maker. However, it can often be difficult for the decision maker to express exact estimates of the ratios of importance of alternatives. This leads us to assume that the preferences over the set of alternatives,

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X, may be represented (at least) in the following two ways: multiplicative and additive. Let us assume that the preferences on X are described by a preference relation on X given by a positive $n \times n$ matrix $A = \{a_{ij}\}$, where $a_{ij} > 0$ for all i, j indicates a preference intensity for alternative x_i to that of x_j, i.e. it is interpreted as "x_i is a_{ij} times as good as x_j". According to [6], T. Saaty suggested measuring a_{ij} using a ratio scale, particularly the scale $\{1/9, 1/8, ..., 1, ..., 8, 9\}$. The elements of $A = \{a_{ij}\}$ satisfy the following reciprocity condition [6].

A positive $n \times n$ matrix $A = \{a_{ij}\}$ is *multiplicative-reciprocal* (*m-reciprocal*), if

$$a_{ij}.a_{ji} = 1 \text{ for all } i, j. \tag{1}$$

A positive $n \times n$ matrix $A = \{a_{ij}\}$ is *multiplicative-consistent* (or, *m-consistent*) [3], [6], if

$$a_{ij} = a_{ik}.a_{kj} \text{ for all } i, j, k. \tag{2}$$

Notice that $a_{ii} = 1$ for all i, and also (2) implies (1), i.e. an m-consistent positive matrix is m-reciprocal (however, not vice-versa). Then, (2) can be rewritten equivalently as

$$a_{ik}.a_{kj}.a_{ji} = 1 \text{ for all } i, j, k. \tag{3}$$

Notice that here $a_{ij} > 0$ and m-consistency is not restricted to the Saatys scale. The above mentioned interpretation of preferences on X described by a positive matrix is, however, not always appropriate for a decision maker. Evaluating the preference of two elements of a pair, say, x_i and x_j with respect to e.g. "design of a product" might cause a problem. Here, saying e.g. that x_i is 3 times as good as x_j is peculiar. Using word categories, e.g. "moderately (strongly, very strongly etc.) better", as it is recommended in AHP [6], is not way out. A more natural way seems to be the following: divide 100% of the property in question into two parts and then assign the first part to the first element and the rest to the second one, see [4]. In other words, when comparing x_i to x_j the decision maker assigns the value b_{ij} to x_i and b_{ji} to x_j, whereas $b_{ij} + b_{ji} = 1$ (i.e. 100%). With this interpretation, the preferences on X can be represented also by a *fuzzy preference relation*, with membership function $\mu_R : X \times X \to [0; 1]$, where $\mu_R(x_i, x_j) = b_{ij}$ denotes the preference of the alternative x_i over x_j [3], [4], [6]. Important properties of the above mentioned matrix $B = \{b_{ij}\}$ can be summarized as follows.

An $n \times n$ matrix $B = \{b_{ij}\}$ with $0 \le b_{ij} \le 1$ for all i and j is *additive-reciprocal* (*a-reciprocal*) [1], if

$$b_{ij} + b_{ji} = 1 \text{ for all } i, j. \tag{4}$$

Evidently, if (4) holds for all i and j, then $b_{ii} = 0.5$ for all i.

For making a coherent choice (when assuming additive fuzzy preference matrices) a set of properties to be satisfied by such relations have been suggested in the literature [3], [7].

Transitivity is one of the most important properties concerning preferences, and it represents the idea that the preference intensity obtained by comparing directly two alternatives should be equal to or greater than the preference intensity "between" those two alternatives obtained using an indirect chain of alternatives. Some of the suggested properties – candidates for consistency property – are given here, see also [1], [7].

Let $B = \{b_{ij}\}$ be an $n \times n$ a-reciprocal matrix with $0 \le b_{ij} \le 1$ for all i and j.

1. *Weak transitivity* [7]:

$$\text{If } b_{ij} \ge 0.5 \text{ and } b_{jk} \ge 0.5 \text{ then } b_{ik} \ge 0.5 \text{ for all } i, j, k. \tag{5}$$

2. *Restricted max-transitivity* [7]:

$$\text{If } b_{ij} \ge 0.5 \text{ and } b_{jk} \ge 0.5 \text{ then } b_{ik} \ge max\{b_{ij}, b_{jk}\} \text{ for all } i, j, k. \tag{6}$$

3. *Multiplicative-transitivity (m-transitivity)* [7]:

$$b_{ij}.b_{jk}.b_{ki} = b_{ik}.b_{kj}.b_{ji} \text{ for all } i, j, k. \tag{7}$$

If $b_{ij} > 0$ for all i and j, then (7) can be rewritten as

$$\frac{b_{ij}}{b_{ji}} \cdot \frac{b_{jk}}{b_{kj}} \cdot \frac{b_{ki}}{b_{ik}} = 1 \text{ for all } i, j, k. \tag{8}$$

It is easy to prove that if $B = \{b_{ij}\}$ is m-transitive, then it is restricted max-transitive. Evidently, the opposite is not true. Notice that if B is m-consistent then B is m-transitive. Moreover, if $B = \{b_{ij}\}$ is m-reciprocal, then B is m-transitive iff B is m-consistent.

4. *Additive-transitivity* (*a-transitivity*) [7]:

$$b_{ij} + b_{jk} + b_{ki} = 1.5 \text{ for all } i, j, k. \tag{9}$$

It is not difficult to prove that if $B = \{b_{ij}\}$ is additive transitive, then it is restricted max-transitive. Therefore, the additive transitivity is stronger than restricted max-transitivity.

3 Additive versus multiplicative-reciprocal matrices

In this section we shall investigate some relationships between a-reciprocal and m-reciprocal pair-wise comparison matrices. We start with extension of the result published by E. Herrera-Viedma et al. [1]. For this purpose, given $\sigma > 1$, we define the following function φ_σ and its inverse function φ_σ^{-1} as

$$\varphi_\sigma(t) = \frac{1}{2}(1 + \frac{\ln t}{\ln \sigma}) \text{ for } t \in [1/\sigma; \sigma], \tag{10}$$

$$\varphi_\sigma^{-1}(t) = \sigma^{2t-1} \text{ for } t \in [0; 1]. \tag{11}$$

We obtain the following results, characterizing a-transitive and m-consistent matrices, see [1].

Proposition 1. Let $A = \{a_{ij}\}$ be an $n \times n$ matrix with $\frac{1}{\sigma} \leq a_{ij} \leq \sigma$ for all i and j.

If $A = \{a_{ij}\}$ is m-consistent then $B = \{\varphi_\sigma(a_{ij})\}$ is a-transitive.

Proposition 2. Let $B = \{b_{ij}\}$ be an $n \times n$ matrix with $0 \leq b_{ij} \leq 1$ for all i and j.

If $B = \{b_{ij}\}$ is a-transitive then $A = \{\varphi_\sigma^{-1}(b_{ij})\}$ is m-consistent.

Now, let us define the function ϕ and its inverse function ϕ^{-1} as follows

$$\phi(t) = \frac{t}{1+t} \text{ for } t > 0, \tag{12}$$

$$\phi^{-1}(t) = \frac{t}{1-t} \text{ for } 0 < t < 1. \tag{13}$$

Proposition 3.

Let $A = \{a_{ij}\}$ be an $n \times n$ matrix with $0 < a_{ij}$ for all i and j.

If $A = \{a_{ij}\}$ is m-consistent then $B = \{b_{ij}\} = \{\phi(a_{ij})\}$ is m-transitive.

Proposition 4.

Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j.

If $B = \{b_{ij}\}$ is m-transitive then $A = \{a_{ij}\} = \{\phi^{-1}(b_{ij})\}$ is m-consistent.

From Proposition 2 it is clear that the concept of m-transitivity plays a similar role for a-reciprocal matrices as the concept of m-consistency does for m-reciprocal matrices. That is why it is reasonable to introduce the following definition: Any $n \times n$ nonnegative a-reciprocal matrix $B = \{b_{ij}\}$ which is m-transitive is called *additively consistent* (*a-consistent*). Then Proposition 4 can be reformulated accordingly.

In practice, perfect consistency/transitivity is difficult to obtain, particularly when measuring preferences on a set with a large number of alternatives.

4 Inconsistency of pair-wise comparison matrices, priority vectors

If for some positive $n \times n$ matrix $A = \{a_{ij}\}$ and for some $i, j, k = 1, 2, ..., n$, multiplicative consistency condition (2) does not hold, then A is said to be *multiplicative-inconsistent* (or, *m-inconsistent*). Eventually, if for some $n \times n$ matrix $B = \{b_{ij}\}$ with $0 \leq b_{ij} \leq 1$ for all i and j, and for some $i, j, k = 1, 2, ..., n$, (7) does not hold, then B is said to be *additive-inconsistent* (or, *a-inconsistent*) . Finally, if for some $n \times n$ matrix $B = \{b_{ij}\}$ with $0 \leq b_{ij} \leq 1$ for all i and j, and for some $i, j, k = 1, 2, ..., n$, (9) does not hold, then B is said to be *additive-intransitive* (or, $a - intransitive$). In order to measure the grade of inconsistency/intransitivity of a given matrix several measurement methods have been proposed in the literature. In AHP, multiplicative reciprocal matrices have been considered, see [6].

As far as additive-reciprocal matrices are concerned, some methods for measuring a-inconsistency/a-intransitivity are proposed in this section. Here, instead of positive matrices we consider preference matrices with nonnegative elements, i.e. some elements are eventually zeros. Measuring inconsitency of such matrix is based on Perron-Frobenius theory which is known in several versions, see [2]. The Perron-Frobenius theorem, see e.g. [2], describes some of the remarkable properties enjoyed by the eigenvalues and eigenvectors of irreducible nonnegative matrices (e.g. positive matrices).

Theorem 1. (Perron-Frobenius) Let A be an irreducible nonnegative square matrix. Then the spectral radius, $\rho(A)$, is a real eigenvalue, which has a positive (real) eigenvector. This eigenvalue called the principal eigenvalue of A is simple (it is not a multiple root of the characteristic equation), and its eigenvector called priority vector is unique up to a multiplicative constant.

The m-consistency of a nonnegative m-reciprocal $n \times n$ matrix A is measured by the *m-consistency index* $I_{mc}(A)$ defined in [6] as

$$I_{mc}(A) = \frac{\rho(A) - n}{n - 1}, \tag{14}$$

where $\rho(A)$ is the spectral radius of A (particularly, *the principal eigenvalue* of A).

Ranking the alternatives in X is determined by the vector of weights $w = (w_1, w_2, ..., w_n)$, with $w_i > 0$, for all $i = 1, 2, ..., n$, such that $\sum_{i=1}^{n} w_i = 1$, satisfying $Aw = \rho(A)w$, is called the (*normalized*) *principal eigenvector of* A, or, *priority vector of* A. Since the element of the priority vector w_i is interpreted as the relative importance of alternative x_i, the alternatives $x_1, x_2, ..., x_n$ in X are ranked by their relative importance. The following important result has been derived in [6].

Theorem 2. If $A = \{a_{ij}\}$ is an $n \times n$ positive m-reciprocal matrix, then $I_{mc}(A) \geq 0$. Moreover, A is m-consistent iff $I_{mc}(A) = 0$.

To provide a (in)consistency measure independently of the dimension of the matrix, n, T. Saaty in [6] proposed the consistency ratio. In order to distinguish it here from the other consistency measures, we shall call it *m-consistency ratio.* This is obtained by taking the ratio I_{mc} to its mean value R_{mc} over a large number of positive m-reciprocal matrices of dimension n, whose entries are randomly and uniformly generated, i.e.

$$CR_{mc} = \frac{I_{mc}}{R_{mc}}. \tag{15}$$

For this consistency measure it was proposed an estimation of 10% threshold of CR_{mc}. In other words, a pair-wise comparison matrix could be accepted (in a DM process) if its m-consistency ratio does not exceed 0.1, see [6]. The m-consistency index I_{mc} has been defined by (14) for m-reciprocal matrices, now, we shall investigate inconsistency/intransitivity also for a-reciprocal matrices. For this purpose we use relations between m-consistent and a-transitive/a-consistent matrices derived in Propositions 1 to 4. Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. We define the *a-consistency index* $I_{ac}(B)$ of $B = \{b_{ij}\}$ as

$$I_{ac}(B) = I_{mc}(A), \text{where } A = \{\phi^{-1}(b_{ij})\}. \tag{16}$$

From (16) we obtain the following result which is parallel to Theorem 2.

Theorem 3. If $B = \{b_{ij}\}$ is an a-reciprocal $n \times n$ fuzzy matrix with $0 < b_{ij} < 1$ for all i and j, then $I_{ac}(B) \geq 0$. Moreover, B is a-consistent iff $I_{ac}(B) = 0$.

Now, we shall deal with measuring a-intransitivity of a-reciprocal matrices. Recall transformation functions φ_σ and φ_σ^{-1} defined by (10), (11), where $\sigma > 1$ is a given value. Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. We define the *a-transitivity index* $I_{at}^\sigma(B)$ of $B = \{b_{ij}\}$ as

$$I_{at}^\sigma(B) = I_{mc}(A_\sigma), \text{where } A_\sigma = \{\varphi_\sigma^{-1}(b_{ij})\}. \tag{17}$$

From (11), (17) we obtain the following result which is parallel to Theorem 2 and 3.

Theorem 4. If $B = \{b_{ij}\}$ is an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j, then $I_{at}^\sigma(B) \geq 0$. Moreover, B is a-transitive iff $I_{at}^\sigma(B) = 0$.

Let $A = \{a_{ij}\}$ be an a-reciprocal $n \times n$ matrix. In (15), the *m-consistency ratio of* A denoted by $CR_{mc}(A)$ is obtained by taking the ratio $I_{mc}(A)$ to its mean value $R_{mc}(n)$ over a large number of randomly and uniformly generated positive m-reciprocal matrices of dimension n, i.e.

$$CR_{mc}(A) = \frac{I_{mc}(A)}{R_{mc}(n)}. \tag{18}$$

The table that gives the function values of $R_{mc}(n)$ can be found e.g. in [6]. Similarly, we define a-consistency ratio and a-transitivity ratio. Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. We define the *a-consistency ratio* CR_{ac} of B as follows

$$CR_{ac}(B) = \frac{I_{ac}(B)}{R_{mc}(n)}. \tag{19}$$

The corresponding priority vector $w^{ac} = (w_1^{ac}, w_2^{ac}, ..., w_n^{ac})$ is given by the characteristic equation $\phi^{-1}(B)w^{ac} = \rho(\phi^{-1}(B))w^{ac}$.

Moreover, given $\sigma > 1$, we define *a-transitivity ratio* CR_{at}^σ of B as

$$CR_{at}^\sigma(B) = \frac{I_{at}^\sigma(B)}{R_{mc}(n)}. \tag{20}$$

The corresponding priority vector $w^{at} = (w_1^{at}, w_2^{at}, ..., w_n^{at})$ is given by $\varphi_\sigma^{-1}(B)w^{at} = \rho(\varphi_\sigma^{-1}(B)w^{at}$.

In practical DM situations a-inconsistency of a positive a-reciprocal pair-wise comparison matrix B is “acceptable” if $CR_{ac}(B) < 0.1$. Also, a-intransitivity of a positive a-reciprocal pair-wise comparison matrix B is “acceptable” if $CR_{at}^\sigma(B) < 0.1$. The final ranking of alternatives is given by the corresponding priority vector.

5 Illustrative example

Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of 4 alternatives. The preferences on X are described by a positive matrix $B = \{b_{ij}\}$,

$$B = \begin{pmatrix} 0.5 & 0.6 & 0.6 & 0.9 \\ 0.4 & 0.5 & 0.6 & 0.7 \\ 0.4 & 0.4 & 0.5 & 0.5 \\ 0.1 & 0.3 & 0.5 & 0.5 \end{pmatrix}. \tag{21}$$

Here, $B = \{b_{ij}\}$ is a-reciprocal and it is a-inconsistent, as it may be directly verified by (7), e.g. $b_{12}.b_{23}.b_{31} \neq b_{32}.b_{21}.b_{13}$. At the same time, B is a-intransitive as $b_{12} + b_{23} + b_{31} = 1.9 \neq 1.5$. We consider $\sigma = 9$ and calculate

$$E = \{\phi^{-1}(b_{ij})\} = \begin{pmatrix} 1 & 1.50 & 1.50 & 9.00 \\ 0.67 & 1 & 1.5 & 2.33 \\ 0.67 & 0.67 & 1 & 1 \\ 0.11 & 0.43 & 1 & 1 \end{pmatrix},$$

$$F = \{\varphi_9^{-1}(b_{ij})\} = \begin{pmatrix} 1 & 1.55 & 1.55 & 5.80 \\ 0.64 & 1 & 1.55 & 2.41 \\ 0.64 & 0.64 & 1 & 1 \\ 0.17 & 0.42 & 1 & 1 \end{pmatrix}.$$

We calculate $\rho(E) = 4.29$, $\rho(F) = 4.15$. By (14), (19) and (20) we obtain $CR_{ac}(B) = 0.11 > 0.1$ with the priority vector $w^{ac} = (0.47; 0.25; 0.18; 0.10)$, which gives the ranking of alternatives $x_1 > x_2 > x_3 > x_4$. Similarly, $CR^9_{at}(B) = 0.056 < 0.1$ with the priority vector $w^{at} = (0.44; 0.27; 0.18; 0.12)$, giving the same ranking of alternatives $x_1 > x_2 > x_3 > x_4$.

As it is evident, a-consistency ratio $CR_{ac}(B)$ is too high that matrix B is considered a-consistent. On the other hand, a-transitivity ratio $CR^9_{at}(B)$ is sufficiently low that matrix B is considered a-transitive. The ranking of alternatives given by both methods remains, however, the same.

6 Conclusion

In this paper we investigated two types of pair-wise comparison matrices as well as the concepts of reciprocity, consistency/inconsistency and transitivity/intransitivity. In the literature, an inconsistency measure, i.e. inconsistency index, is known only for m-reciprocal matrix. Here we definedthe inconsistency index also for a-reciprocal matrices. As it was shown, the proposed concepts can be applied in ranking alternatives as well as in eliciting criteria weights in MCDM problems. New inconsistency/intransitivity indices will measure the quality of proposed ranking procedure. Numerical experiments show that there is no strong relationship between a-consistency and a-transitivity.

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