Fuzzy linear programming duality

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Abstract. The word "duality" has been used in various areas of science for long time. Nevertheless, in general, there is a lack of consensus about the exact meaning of this important notion. However, in the field of optimization, and particularly in linear programming, the notion of duality is well understood and remarkably useful. Various attempts to develop analogous useful duality schemes for linear programming involving fuzzy data have been appearing since the early days of fuzzy sets. After recalling basic results on linear programming duality, we give examples of early attempts in extending duality to problems involving fuzzy data, and then we discuss recent results on duality in fuzzy linear programming and their possible application.

Keywords: linear programming, fuzzy linear programming, duality.

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1 Introduction

As pointed out by Harold Kuhn [3] the elements of duality in optimization are: (i) A pair of optimization problem based on the same data, one a maximum problem with objective function $x \mapsto f(x)$ and the other a minimum problem with objective function $y \mapsto h(y)$. (ii) For feasible solutions x and y to the pair of problems, always $h(y) \geq f(x)$. (iii) Necessary and sufficient condition for optimality of feasible solutions $\bar{x}$ and $\bar{y}$ is $h(\bar{y}) = f(\bar{x})$.

This kind of duality is particularly clear, elegant, and remarkably useful in linear programming and its applications. Given the practical relevance of duality theory of linear programming, it is not surprising that attempts to develop analogous duality schemes for linear programming involving fuzzy data have been appearing since the early days of fuzzy sets [8]. To devise such a duality scheme, we have to specify in advance some class of permitted fuzzy numbers, define fundamental arithmetic operations with fuzzy numbers, and clarify the meaning of inequalities between fuzzy numbers. Because this can be done in inexhaustibly many ways, we can hardly expect a unique extension of duality to fuzzy situations, which would be so clean and clear like that of classical linear programming. Instead, there exist several variants of the duality theory for fuzzy linear programming, the results of which resemble in various degrees some of the useful results established in the conventional linear programming.

After recalling basic results of duality theory of linear programming, we first present early examples of pairs of mutually dual problems, in which only the inequalities $\leq$ and $\geq$ are allowed to become fuzzy. The feasible solutions of such problems are nonnegative vectors of a finite dimensional real vector space, and the degrees of constraints satisfaction and the degrees of optimality of feasible solutions are defined by the numerical data from the underlying linear programming problem and valued extensions of binary relations $\leq$ and $\geq$. Then we discuss duality pairs for problems in which some or all numerical data may also be fuzzy. The duality schemes for such problems are significantly more complicated because of necessity to extend $\leq$ and $\geq$ so that some consistent comparison of fuzzy quantities is possible. For reader's convenience of this extended abstract, we summarized necessary notions and results from the theory of fuzzy sets in the Appendix.

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2 Linear Programming Duality

Given real numbers $b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_n, a_{11}, a_{12}, \ldots, a_{mn}$, we consider linear programming problems in the canonical form:

$$\begin{array}{llcll}
\text{Maximize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n & & & (1)\\
\text{subject to} & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i & & i = 1, 2, \ldots, m & (2)\\
 & x_j \geq 0 & & j = 1, 2, \ldots, n & (3)
\end{array}$$

Using the same data $b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_n, a_{11}, a_{12}, \ldots, a_{mn}$, we construct another linear programming problem, called the *dual* problem to the *primal* problem (1)-(3), as follows:

$$\begin{array}{llcll}
\text{Minimize} & y_1b_1 + y_2b_2 + \cdots + y_mb_m & & & (4)\\
\text{subject to} & y_1a_{1j} + y_2a_{2j} + \cdots + y_ma_{mj} \geq c_j & & j = 1, 2, \ldots, n & (5)\\
 & y_i \geq 0 & & i = 1, 2, \ldots, m & (6)
\end{array}$$

It is easy to see that if one rewrites the dual problem into the form of the primal problem and again constructs the corresponding dual, then one obtains a linear programming problem which is equivalent to the original primal problem. In other words, the dual to the dual is the primal. Consequently, it is just the matter of convenience which of these problems is taken as the primal problem.

The well known results on the mutual relationships between the primal and the dual can be summarized as follows:

1. If x is a feasible solution of the primal problem and if y is a feasible solution of the dual problem, then $cx \leq yb$.
2. If $\bar{x}$ is a feasible solution of the primal problem, and if $\bar{y}$ is a feasible solution of the dual problem, and if $c\bar{x} = \bar{y}b$, then $\bar{x}$ is optimal for the primal problem and $\bar{y}$ is optimal for the dual problem.
3. If the feasible region of the primal problem is nonempty and the objective function $x \mapsto cx$ is not bounded above on it, then the feasible region of the dual problem is empty.
4. If the feasible region of the dual problem is nonempty and the objective function $y \mapsto yb$ is not bounded below on it, then the feasible region of the primal problem is empty.

It turns out that the following deeper results concerning mutual relation between the primal and dual problems hold:

5. If either of the problems (1)-(3) or (4)-(6) has an optimal solution, so does the other, and the corresponding values of the objective functions are equal.
6. If both of the problems (1)-(3) and (4)-(6) have feasible solutions, then both of them have optimal solutions and the corresponding optimal values are equal.
7. A necessary and sufficient condition that feasible solutions x and y of the primal and dual problems are optimal is that

$$\begin{array}{rcll}
x_j > 0 & \Rightarrow & yA^j = c_j & 1 \leq j \leq n\\
x_j = 0 & \Leftarrow & yA^j > c_j & 1 \leq j \leq n\\
y_i > 0 & \Rightarrow & A_ix = b_i & 1 \leq i \leq m\\
y_i = 0 & \Leftarrow & A_ix < b_i & 1 \leq i \leq m
\end{array}$$

where A^j and A_i stand for the j-th column and i-th row of $A = \{a_{ij}\}$, respectively.

It is also well known that the essential duality results of linear programming can be expressed as a saddle-point property of the Lagrangian function:

8. Let $\mathbb{R}^n_+$ and $\mathbb{R}^m_+$ denote the set of nonnegative n-vectors and m-vectors, and let $L\colon \mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}$ be the *Lagrangian function* for the primal problem (1)-(3), that is, $L(x, y) = cx + y(b - Ax)$. The necessary and sufficient condition that $\bar{x} \in \mathbb{R}^n_+$ be an optimal solution of the primal problem (1)-(3) and $\bar{y} \in \mathbb{R}^m_+$ be an optimal solution of the dual problem (4)-(6) is that $(\bar{x}, \bar{y})$ be a saddle point of L; that is, for all $x \in \mathbb{R}^n_+$ and $y \in \mathbb{R}^m_+$,

$$L(x, \bar{y}) \leq L(\bar{x}, \bar{y}) \leq L(\bar{x}, y) \tag{7}$$

3 Dual Pairs of Rödder and Zimmermann

One of the early approaches to duality in linear programing problems involving fuzziness is due to Rödder and Zimmermann [8]. To be able to state the problems considered by Rödder and Zimmermann concisely, we first observe that the conditions (7) bring up the pair of optimization problems

$$\text{maximize} \quad \min_{y \geq 0} L(x, y) \quad \text{subject to } x \in \mathbb{R}^n_+ \tag{8}$$

$$\text{minimize} \quad \max_{x \geq 0} L(x, y) \quad \text{subject to } y \in \mathbb{R}^m_+ \tag{9}$$

Let μ and μ' be real valued functions on $\mathbb{R}^n_+$ and $\mathbb{R}^m_+$, respectively; and let $\{\nu_x\}_{x \in \mathbb{R}^n_+}$ and $\{\nu'_y\}_{y \in \mathbb{R}^m_+}$ be families of real valued functions on $\mathbb{R}^m_+$ and $\mathbb{R}^n_+$, respectively. Furthermore, let φ_y and ψ_x be real valued functions on $\mathbb{R}^n_+$ and $\mathbb{R}^m_+$ defined by $\varphi_y(x) = \min(\mu(x), \nu_x(y))$ and $\psi_x(y) = \min(\mu'(y), \nu'_y(x))$. Now let us consider the following pair of families of optimization problems:

Family $\{P_y\}$:	Given $y \in \mathbb{R}^m_+$, maximize $\varphi_y(x)$ subject to $x \in \mathbb{R}^n_+$
Family $\{D_x\}$:	Given $x \in \mathbb{R}^n_+$, maximize $\psi_x(y)$ subject to $y \in \mathbb{R}^m_+$

Motivated and supported by economic interpretation, Rödder and Zimmermann [8] propose to specify functions μ and μ' and families $\{\nu_x\}$ and $\{\nu'_y\}$ as follows: Given an $m \times n$ matrix A, $m \times 1$ vector b, $1 \times n$ vector c, and real numbers γ and δ, define the functions μ, μ', ν_x and ν'_y by

$$\mu(x) = \min(1, 1 - (\gamma - cx)), \quad \mu'(y) = \min(1, 1 - (yb - \delta)) \tag{10}$$

$$\nu_x(y) = \max(0, y(b - Ax)), \quad \nu'_y(x) = \max(0, (yA - c)x) \tag{11}$$

Strictly speaking, we do not obtain a duality scheme as conceived by Kuhn because there is no relationship between the numbers γ and δ. Indeed, if the the family $\{P_y\}_{y \geq 0}$ is considered to be the primal problem, then we have the situation in which the primal problem is completely specified by data A, b, c and γ. However, these data are not sufficient for specification of family $\{D_x\}_{x \geq 0}$ because the definition of $\{D_x\}_{x \geq 0}$ requires knowledge of δ. Thus from the point of view that the dual problem is to be constructed only on the basis of the primal problem data, every choice of δ determines a certain family dual to $\{P_y\}_{y \geq 0}$. In this sense we could say that every choice of δ gives a duality, the δ-*duality*. Analogously, if the primal problem is $\{D_x\}_{x \geq 0}$, then every choice of γ determines some family $\{P_y\}_{y \geq 0}$ dual to $\{D_x\}_{x \geq 0}$, and we obtain the γ-*duality*. In other words, for every γ, δ, we obtain (γ, δ)-duality. It is worth noticing that families $\{P_y\}$ and $\{D_x\}$ consist of uncountably many linear optimization problems. Moreover, every problem of each of these families may have uncountably many optimal solutions. Consequently, the solution of the problem given by family $\{P_y\}_{y \geq 0}$ is the family $\{X(y)\}_{y \geq 0}$ of subsets of $\mathbb{R}^n_+$ where $X(y)$ is the set of maximizers of φ_y over $\mathbb{R}^n_+$. Analogously, the family $\{Y(x)\}_{x \geq 0}$ of maximizers of ψ_x over $\mathbb{R}^m_+$ is the solution of problem given by family $\{D_x\}_{x \geq 0}$. Rödder and Zimmermann propose to replace the families $\{P_y\}$ and $\{D_x\}$ by the families $\{P'_y\}$ and $\{D'_x\}$ of problems defined as follows:

$$\text{maximize} \quad \lambda \quad \text{subject to } \lambda \leq 1 + cx - \gamma, \ \lambda \leq y(b - Ax), \ x \geq 0 \tag{12}$$

$$\text{minimize} \quad \eta \quad \text{subject to } \eta \geq yb - \delta - 1, \ \eta \geq (c - yA)x, \ y \geq 0 \tag{13}$$

They call these families of optimization problems the *fuzzy dual pair* and claim that the families $\{P_y\}$ and $\{D_x\}$ become families $\{P'_y\}$ and $\{D'_x\}$ when μ, μ', ν_x and ν'_y are defined by (10)-(11). To see that this claim cannot be substantiated, it suffices to observe that the value of function φ_y cannot be greater than 1, whereas the value of λ is not bounded above whenever A and b are such that both cx and $-yAx$ are positive for some $x \in R^n_+$. To obtain a valid conversion, one needs to add the inequalities $\lambda \leq 1$ and $\eta \geq -1$ to the constraints. Thus it seems that more suitable choice of functions ν_x and ν'_y in the Rödder and Zimmermann duality scheme would be $\nu_x(y) = \min(1, 1+y(b-Ax))$ and $\nu'_y(x) = \min(1, 1+(yA-c)x)$. Another objection to the Rödder and Zimmermann model arises from the fact that, the duality results for the proposed fuzzy dual pair do not reduce to the standard duality results for the crisp scenario, that is, for $\lambda = 1, \eta = -1$. Again an easy remedy is to work with $\nu_x(y) = \min(1, 1 + y(b - Ax))$ and $\nu'_y(x) = \min(1, 1+(yA-c)x)$ instead of $\nu_x(y) = \min(0, 1+y(b-Ax))$ and $\nu'_y(x) = \min(0, 1+(yA-c)x)$.

4 Duality Pairs of Bector and Chandra

In contrast to the usual practice, in the Rödder and Zimmermann model, the range of membership functions μ and μ' is $(-\infty, 1]$, and the range of membership functions ν_x and ν'_u is $[0, \infty)$ or $[1, \infty)$ instead of usual $[0, 1]$. Bector and Chandra [1] propose to replace the relations $\leq$ and $\geq$ appearing in the dual pair of linear programming problems by their valued extensions. In particular, the inequality $\leq$ appearing in the ith constraint of the primal problem is replaced by its valued extension $\preceq_i$ whose membership function $\mu_{\preceq_i}\colon \mathbb{R} \times \mathbb{R} \to [0, 1]$ is defined by

$$\mu_{\preceq_i}(\alpha, \beta) = \begin{cases} 1 & \text{if} \quad \alpha \leq \beta \\ 1 - \frac{\alpha-\beta}{p_i} & \text{if} \quad \beta < \alpha \leq \beta + p_i \\ 0 & \text{if} \quad \beta + p_i < \alpha \end{cases}$$

where p_i is a positive number. Analogously, the inequality $\geq$ appearing in the jth constraint of the dual problem is replaced by its valued relation $\succeq_j$ with membership function

$$\mu_{\succeq_j}(\alpha, \beta) = \begin{cases} 1 & \text{if} \quad \alpha \geq \beta \\ 1 - \frac{\beta-\alpha}{q_j} & \text{if} \quad \beta > \alpha \geq \beta - q_j \\ 0 & \text{if} \quad \beta - q_j > \alpha \end{cases}$$

where q_j is a positive number. The degree of satisfaction with which $x \in \mathbb{R}^n$ fulfills the ith fuzzy constraint $A_i x \preceq_i b_i$ of the primal problem is expressed by the fuzzy subset of $\mathbb{R}^n$ whose membership function μ_i is defined by $\mu_i(x) = \mu_{\preceq_i}(A_i x, b_i)$, and the degree of satisfaction with which $y \in \mathbb{R}^m$ fulfills the jth fuzzy constraint $yA^j \succeq_j c_j$ of the dual problem is expressed by the fuzzy subset of $\mathbb{R}^m$ whose membership function μ_j is defined by $\mu_j(y) = \mu_{\succeq_j}(yA^j, c_j)$. Similarly, we can express the degree of satisfaction with a prescribed aspiration level γ of the objective function value cx by the fuzzy subset of $\mathbb{R}^n$ given by $\mu_0(x) = \mu_{\succeq_0}(cx, \gamma)$ where, for the tolerance given by a positive number p_0, the membership function $\mu_{\succeq_0}$ is defined by

$$\mu_{\succeq_0}(\alpha, \beta) = \begin{cases} 1 & \text{if} \quad \alpha \geq \beta \\ 1 - \frac{\beta-\alpha}{p_0} & \text{if} \quad \beta > \alpha \geq \beta - p_0 \\ 0 & \text{if} \quad \beta - p_0 > \alpha \end{cases}$$

Analogously, for the degree of satisfaction with the aspiration level δ and tolerance q_0 in the dual problem, we have $\mu_0(y) = \mu_{\preceq_0}(\delta, yb)$ where

$$\mu_{\preceq_0}(\alpha, \beta) = \begin{cases} 1 & \text{if} \quad \alpha \leq \beta \\ 1 - \frac{\alpha-\beta}{q_0} & \text{if} \quad \beta < \alpha \leq \beta + q_0 \\ 0 & \text{if} \quad \beta + q_0 < \alpha \end{cases}$$

This leads to the following pair of linear programming problems:

Given positive numbers $p_0, p_1, \ldots, p_m$, and a real number γ, maximize λ subject to

$$\begin{aligned} &(\lambda - 1)p_0 \leq cx - \gamma \\ &(\lambda - 1)p_i \leq b_i - A_i x, \; 1 \leq i \leq m \\ &0 \leq \lambda \leq 1, x \geq 0 \end{aligned} \tag{14}$$

Given positive numbers $q_0, q_1, \ldots, q_n$, and a real number δ, minimize $-\eta$ subject to

$$\begin{aligned} &(\eta - 1)q_0 \leq \delta - yb \\ &(\eta - 1)q_j \leq yA^j - c_j, \; 1 \leq j \leq n \\ &0 \leq \eta \leq 1, y \geq 0 \end{aligned} \tag{15}$$

Bector and Chandra call this pair the *modified fuzzy pair* of primal dual linear programming problems. Again we see that the dual problem is not stated by using only the data available in the primal problem. Indeed, if the problem (14) is considered to be the primal problem, then to state its dual problem one needs additional information; namely, a number δ and numbers $q_0, q_1, \ldots, q_n$; if problem (15) is considered to be primal, then one needs a number γ and numbers $p_0, p_1, \ldots, p_m$.

5 Duality Pairs of Ramík

As mentioned in the Introduction, when we wish to develop a sensible duality scheme for linear programing problems in which also the numerical data may be fuzzy, then we need tools for comparing fuzzy numbers. Recently, Ramík [6] and [5] (see also [2]) proposed a rather general duality scheme in which the fuzzy quantities are compared by means of extensions of binary relations $\leq$ and $\geq$ on $\mathbb{R}$ to fuzzy relations on $\mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R})$. Moreover, this scheme does not require external specification of goals or aspiration levels, and a number of earlier duality schemes can be obtained as special cases. A simple version of this scheme can briefly be described as follows.

Given fuzzy numbers $B_1, B_2, \ldots, B_m; C_1, C_2, \ldots, C_n; A_{11}, A_{12}, \ldots, A_{mn}$ from some class of fuzzy numbers, and fuzzy extensions $\preceq_1, \ldots, \preceq_m; \succeq_1, \ldots, \succeq_n$ of $\leq$ and $\geq$, respectively, we construct the pair of problems

$$\begin{array}{llcll} \text{Maximize} & C_1x_1 + C_2x_2 + \cdots + C_nx_n & & & \quad (16) \\ \text{subject to} & A_{i1}x_1 + A_{i2}x_2 + \cdots + A_{in}x_n & \preceq_i & B_i \quad i = 1, 2, \ldots, m & \quad (17) \\ & x_j & \geq & 0 \quad j = 1, 2, \ldots, n & \quad (18) \end{array}$$

$$\begin{array}{llcll} \text{Minimize} & y_1B_1 + y_2B_2 + \cdots + y_mB_m & & & \quad (19) \\ \text{subject to} & y_1A_{1j} + y_2A_{2j} + \cdots + y_mA_{mj} & \succeq_j & C_j \quad j = 1, 2, \ldots, n & \quad (20) \\ & y_i & \geq & 0 \quad i = 1, 2, \ldots, m & \quad (21) \end{array}$$

where "+" is defined by the standard extension principle, and where the meanings of "feasibility" and "optimality" are specified as follows.

Let β be a positive number from $[0, 1]$. By a *β-feasible region* of problem (16)-(18) we understand the β-cut of fuzzy subset X of $\mathbb{R}^n$ given by membership function

$$\mu_X(x) = \begin{cases} \min\limits_{1 \leq i \leq m} \mu_{\preceq_i}(A_{i1}x_1 + \cdots + A_{in}x_n, B_i) & \text{if } x_j \geq 0 \text{ for all } j \\ 0 & \text{otherwise} \end{cases} \quad (22)$$

and by a *β-feasible solutions* of problem (16)-(18) we understand the elements of *β-feasible region*.

To explain "maximization", we first observe that a feasible solution $\bar{x}$ of non fuzzy problem (1)-(3) is optimal exactly when there is no feasible solution x such that $cx > c\bar{x}$. This suggests to consider a fuzzy extension $\succeq$ of $\geq$ and to introduce, for each positive α from $[0, 1]$, the binary relations $\geq_\alpha$ and $<_\alpha$ on $\mathcal{F}(\mathbb{R})$ by "$a \geq_\alpha b$" means $\mu_\succeq(a, b) \geq \alpha$, and "$a <_\alpha b$" means ($\mu_\succeq(a, b) \geq \alpha$ and $\mu_\succeq(b, a) < \alpha$). Now let α and β be positive numbers from $[0, 1]$. We say that a β-feasible solution $\bar{x}$ of (16)-(18) is (α, β)-*maximal solution* of (16)-(18) if there is no β-feasible solution of (16)-(18) x different from $\bar{x}$ such that $C_1\bar{x}_1 + C_2\bar{x}_2 + \cdots + C_n\bar{x}_n <_\alpha C_1x_1 + C_2x_2 + \cdots + C_nx_n$. The notions of β-feasibility and (α, β)-minimality for the dual problem (19)-(21) are defined analogously.

6 Appendix

Let $\mathcal{U}$ be a fixed nonempty set and let X, Y and Z be subsets of $\mathcal{U}$. Recall that if f is a function from Y to Z and X is a subset of Y, then the function g from X to Z such that $g(x) = f(x)$ for all $x \in X$ is called the *restriction* of f to X, and f is called an *extension* of g to Y. If A is a subset of X, then the *characteristic function* of A is the function χ_A from X to $\{0, 1\}$ such that $\chi_A(x) = 1$ for $x \in X$ and $\chi_A(x) = 0$ otherwise.

The phrase "membership function of a fuzzy set" is very common one in the fuzzy set literature. Obviously such a phrase and similar ones strongly suggest that "fuzzy sets" and "membership functions of fuzzy sets" are different objects. We follow the opinion that fuzzy sets are special nested families of subsets of a set. In more detail (for full details, see [4] or [7]), a *fuzzy subset* A of X is the family $\{A_\alpha\}_{\alpha\in[0,1]}$ of subsets of X such that $A_0 = X, A_\beta \subset A_\alpha$ whenever $0 \leq \alpha \leq \beta \leq 1$, and $A_\beta = \cap_{0\leq\alpha<\beta} A_\alpha$ whenever $0 < \beta \leq 1$. If A is a fuzzy subset of X, then the *membership function of* A is the function $\mu_A : X \to [0,1]$ defined by $\mu_A(x) = \sup\{\alpha : x \in A_\alpha\}$, and the function value $\mu_A(x)$ is called the *membership degree* of x in A. For each $\alpha \in [0,1]$, the set $\{x \in X \mid \mu_A(x) \geq \alpha\}$ is called the *α-cut of* A. It is worth noting that if f is an arbitrary function from X into $[0,1]$, then the family $A = \{A_\alpha\}_{\alpha\in[0,1]}$ of sets $\{x \in X \mid f(x) \geq \alpha\}$ is a fuzzy subset of X and f is a membership function of A. Moreover, if μ_A is the membership function of a fuzzy subset of A, then the α-cut of A coincides with A_α for each $\alpha \in [0,1]$. Therefore, there is a natural one-to-one correspondence between fuzzy subsets of X and real functions from X to $[0,1]$, and each fuzzy subset A of X can be specified by its membership function μ_A and vice-versa. Consequently, it does not matter whether we introduce or discuss the properties of fuzzy subsets of a set in terms of families subsets or membership functions, and the meaning of phrases like "the fuzzy subset of X determined by a membership function $\mu : X \to [0,1]$" or "the fuzzy set $\mu : X \to [0,1]$" becomes clear. Because of the existence of one-to-one correspondence between the subsets and characteristic functions of subsets and because there is also a one-to-one correspondence between the characteristic functions of subsets and the membership functions with values in $\{0,1\}$, we can view subsets of X as fuzzy subsets of X. When we need to distinguish the latter from the other fuzzy subsets of X, we call them the *crisp fuzzy subsets* of X. We denote the collection of all fuzzy subsets of X by $\mathcal{F}(X)$ and the collection of all crisp fuzzy subsets of X by $\mathcal{P}(X)$.

If A is from $\mathcal{F}(X)$, then the set $\{x \in X : \mu_A(x) = 1\}$ is called the *core* of A. If B is from $\mathcal{F}(X)$ and A is from $\mathcal{P}(X)$, and if $\mu_B(x) = \mu_A(x)$ for all x in the core of A, then we say that B is a *valued extension* of A. Because of the existence of one-to-one correspondence between subsets of X and the elements of $\mathcal{P}(X)$, we also have *valued extensions of subsets* of X. Recall that subsets of $X \times X$ are called *binary relations* on X and fuzzy subsets of $X \times X$ are called *fuzzy relation* on X. Applying the previous construction to $X \times X$, we obtain valued extensions of crisp fuzzy relations on X and valued extensions of binary relations on X. Finally, by similar construction we can obtain a valued extensions of fuzzy relations on X to fuzzy relations on $\mathcal{F}(X)$.

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References

[1] Bector, C. R., and Chandra, C.: On duality in linear programming under fuzzy environment, *Fuzzy Sets and Systems* **125** (2002), 317–325.

[2] Inuiguchi, M., Ramík, J., Tanino, T., and Vlach, M.: Satisficing solutions and duality in interval and fuzzy linear programming, *Fuzzy sets and systems* **135** (2003), 151–177.

[3] Kuhn, H. W.: Nonlinear Programming: A Historical View. In: *SIAM–AMS Proceedings, Volume 9*, AMS, Providence, Rhode Island, 1976, 1–26.

[4] D. A. Ralescu, D. A.: A generalization of the representation theorem, *Fuzzy Sets and Systems* **51** (1992), 309–311.

[5] Ramík, J.: Duality in fuzzy linear programming: Some new concepts and results, *Fuzzy Optimization and Decision Making* **4** (2005), 25–39.

[6] Ramík, J.: Duality in fuzzy linear programming with possibility and necessity relations, *Fuzzy Sets and Systems* **157** (2006), 1283–1302.

[7] Ramík, J., and Vlach, M.: A non-controversial definition of fuzzy sets. In *Transactions on rough sets II - Rough sets and fuzzy sets.* (J. F. Peters, and A. Skowron, eds.), Springer-Verlag, Berlin, Heidelberg, 2004, 201–207.

[8] Rödder, W., and Zimmermann, H-J.: Duality in fuzzy linear programming. In: *Extremal Methods and System Analysis* (A. V. Fiacco, and K. O. Kortanek, eds.), Berlin, New York, 1980, 415–429.