

# Faster convergence for estimates of parameters of Gompertz-Makeham function using available methods in solver MS Excel 2010

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**Abstract.** In basic applications of specific mortality rates we use moving averages of different lengths and different set of weights. Given that the weighted MAs aren't enough, it's useful to choose one of the sophisticated functions. One of these functions is Gompertz-Makeham and also modified G-M, which are both of exponential type. The initial estimates of parameters can be set up manually and using by other software it's possible to refine it. Given that the optimization SW is relatively expensive, it's useful MS Excel Solver, which is able to solve simple LP and NLP tasks within a reasonable time. Even in v. 2007 there was available basic method to solve simple LP and NLP only, and the additional package was necessary to purchase externally. Since v. 2010 is incorporated in standard package GRG-NL and Evolutionary method, which solve more difficult NLP in less time than before. A subsequent study will provide a comparison of computational time needed to obtain accurate estimates of parameters of functions in solvers v. 2007 and 2010, based on real data of selected EU countries. If is possible to refine the initial parameters of this function, then using solver 2010 and in less time.

**Keywords:** Gompertz-Makeham, MS Excel 2010, specific death rates, convergence.

**JEL Classification:** C61, C63

**AMS Classification:** 90C30

## 1 Introduction

Optimization approaches are also important in demographics, because for special types of applications it is necessary to estimate and refine the parameters of specific functions. One of these function is Gompertz-Makeham function for balancing specific death rates (see Gompertz [6], Makeham [10]), which is used in the construction of complete life tables for commercial use (see e.g. the projection of the human capital of the Czech Republic and its regions to 2050 by Fiala et al. [5] or the development of life expectancy in the Czech Republic and EU countries by Langhamrová Jana and Langhamrová Jitka [9]). This function is of exponential type and therefore is effective use a special procedure for its solution. The initial estimates of its parameters can be calculated from empirical data (see Cipra [1]) and using the optimization software is also possible to improve these estimates. Because the optimization software is expensive and in comparison with other applications of operations research, the balancing of specific death rates is relatively simple application, it is not effective to invest in making professional software, but it is better to use available solutions in a readily available application, e.g. MS Office.

The solver in MS Excel 2007 was equipped with only basic module for solving linear programming and simple nonlinear programming. Additional modules for solving more difficult problems were only available after the external purchase. This has raised the question of whether is better to buy optimization software directly than a supplement to Microsoft Office suite. With the coming of version of MS Excel 2010 the modules GRG Nonlinear and Evolutionary have been incorporated into the solver, which can be used to solve more difficult nonlinear problems. These modules are available from the 2010 release in the basic installation of the base price, so there is not necessary to re-buy it externally.

## 2 Modeling of specific death rates

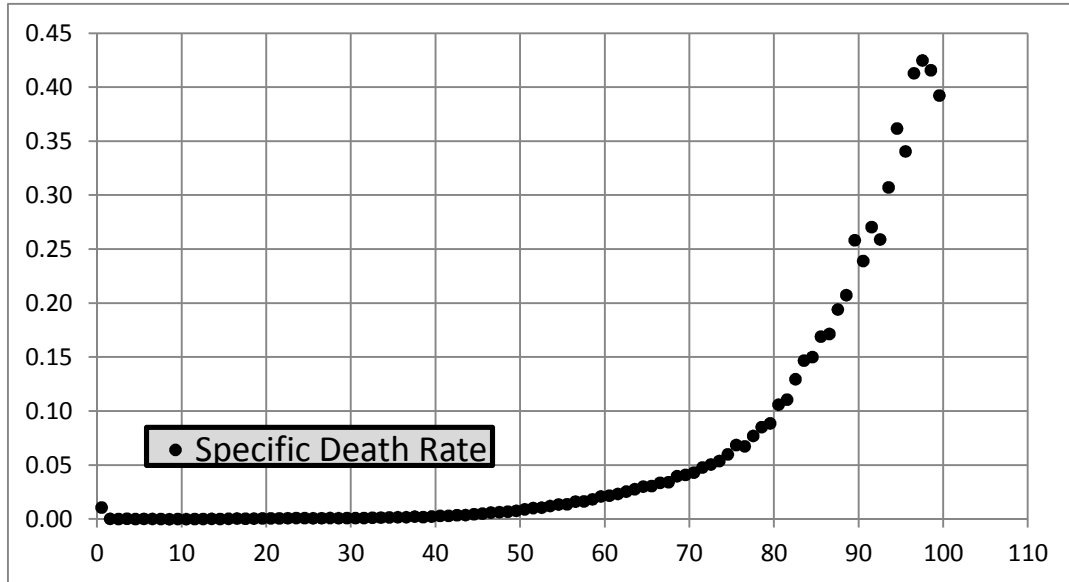
In demographic analysis we use the modeling of specific death rates particularly for the purpose of life insurance (see Cipra [2] or Fiala [3]). The life tables describe the dependence of mortality on the age. The basic characteristics for their calculation are therefore specific death rates. In the case of life tables for one calendar year  $t$  (see Fiala [4]) are specific death rates calculated by the formula

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$$m_x = \frac{M_{t,x}}{\frac{S_{t,x} + S_{t+1,x}}{2}}, \quad (1)$$

where  $M_{t,x}$  is the number of death in the age of  $x$  in year  $t$ ,  $S_{t,x}$  is the initial state of  $x$ -years old person in year  $t$ . The development of specific death rates for the Czech Republic in 2000 is shown for illustration in Figure 1 (data source: Eurostat, own construction) and the history of mortality in selected European countries can be found in [8]).



**Figure 1** The specific death rates for the Czech Republic in 2000.

Death rates are developed up to particular age  $x$  (usually  $x = 60$ ) by previously expected system. For different types of populations are these values more or less scattered in higher age, and for the purpose of further analysis is necessary to balance them. This balancing is performed using either a weighted moving averages (simple method), or with sophisticated balancing function (complex method). High quality outputs provides Gompertz-Makeham balancing function for specific death rates (see Koschin [7]), defined as

$$\tilde{m}_x^{(GM)} = a + b \cdot c^{x+\frac{1}{2}}, \quad (2)$$

for which we set up the beginning of the first age interval  $x_0 = 60$  and the length of interval  $k = 8$ . The initial estimations of parameters of Gompertz-Makeham function is selected according to the recommendations (see Fiala [4]) the following algorithm:

**Algorithm 1.** Calculate the empirical sums of specific death rates in the various intervals as

$$G_1 = \sum_{x=60}^{67} m_x, \quad (3)$$

$$G_2 = \sum_{x=68}^{75} m_x \quad (4)$$

and

$$G_3 = \sum_{x=76}^{83} m_x. \quad (5)$$

Next calculate  $c^k$  and  $c$  (where  $k = 8$ ) as

$$c^8 = \frac{G_3 - G_2}{G_2 - G_1}, \quad (6)$$

$$c = \sqrt[8]{c^8}. \quad (7)$$

Calculate the value of the auxiliary expression

$$K_c = c^{60.5} \cdot (1 + c + \dots + c^7) = c^{60.5} \cdot \frac{c^8 - 1}{c - 1}, \quad (8)$$

and then

$$b = \frac{G_2 - G_1}{K_c \cdot (c^8 - 1)}, \quad (9)$$

$$a = \frac{G_1 - b \cdot K_c}{8}. \quad (10)$$

The values of Gompertz-Makeham function (2) we calculated for the values of age from 60 to 83 years. As the values of the parameters  $a$ ,  $b$  and  $c$  we use the values, that will refine the least squares method. For the age interval  $x \in \langle 83; 110 \rangle$  we use instead of the original Gompertz-Makeham function its modification (see Koschin [7]), defined as

$$\tilde{m}_x^{(GM)'} = a + b \cdot c^{83 + [\ln(d \cdot (x + \frac{1}{2}) - 83) + 1] / d}. \quad (11)$$

As an initial estimate of the parameter  $d$  we choose for example  $d = 0.02$ . Next, for age 83 we overwrite the value given by Gompertz-Makeham function (2) with modified Gompertz-Makeham function given by formula (11). The value of  $d$  will be optimized using least squares method. Weighted squares of deviations, which are defined as

$$\frac{S_{t,x} + S_{t+1,x}}{2 \cdot m_x \cdot (1 - m_x)} \cdot (m_x - \tilde{m}_x^{(GM)'})^2, \quad (12)$$

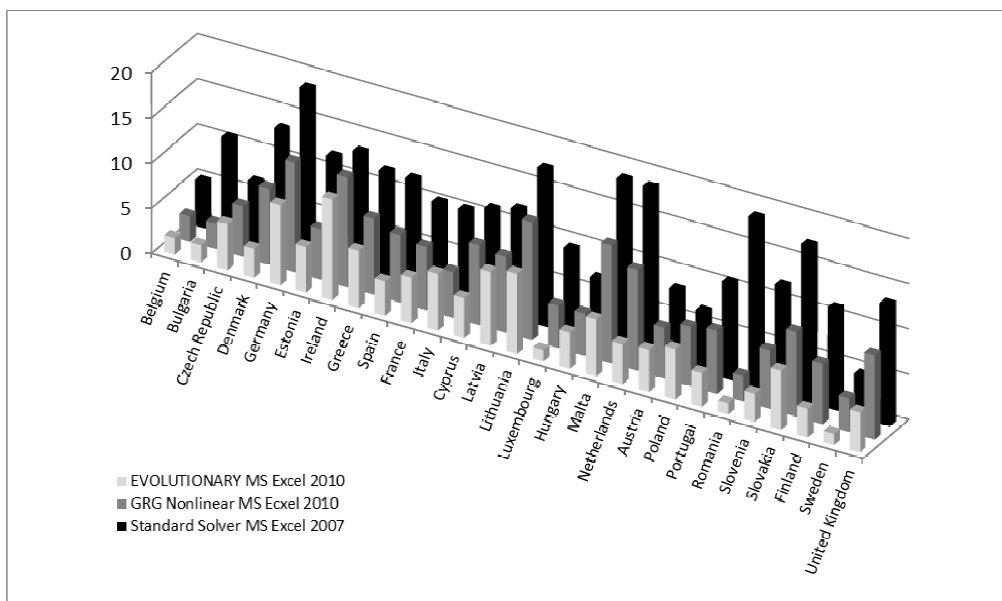
we calculate for all values of  $x$  from age 60 to  $y$ , where  $y$  is the maximum age for which we have empirical death rate with non-zero value. Weighted sums of squared deviations we calculate separately for the age in range 60–82 years (the sum of  $S_1$ ), where we use the original (unmodified) function, and particularly for the age in range of 83– $y$  (sum of  $S_2$ ), where we use a modified function. Optimization procedure is performed with the "solver", implemented in MS Excel. Optimization is performed twice: first separately for sum of squares  $S_1$  (when we change the parameters  $a$ ,  $b$  and  $c$ ) and separately for the second sum of squares  $S_2$  when we change only the parameter  $d$ . Because we used the empirical values of specific death rates to estimate the weights, we do not have to insert any limiting conditions.

### 3 Results and comparison of computational times

For each country of the European Union, the procedure was repeated ten times with the concrete solver and the computing times were measured. Ten times the measurements were taken in a standard module in solver of MS Excel 2007, ten times the measurements were taken in the module GRG Nonlinear in solver in MS Excel 2010, and finally ten times the measurements were taken in the module Evolutionary in solver in MS Excel 2010. From these times there was calculated the average time, that has been necessary for the calculation of the corrected estimations of parameters of Gompertz-Makeham function and modified Gompertz-Makeham function. The used CPU was Intel ® Core™ i5-2430 2.40 GHz running at 64-bit platform. An overview of these computational times is summarized in Table 1.

Country	Standard Solver MS Excel 2007	GRG Nonlinear MS Excel 2010	EVOLUTIONARY MS Excel 2010
Belgium	5.048	2.793	1.862
Bulgaria	10.648	2.793	1.893
Czech Republic	6.655	5.586	5.048
Denmark	13.31	8.379	3.155
Germany	18.634	12.103	8.834
Estonia	11.979	5.586	5.048
Ireland	13.31	12.103	11.172
Greece	11.979	8.379	6.31
Spain	11.979	7.448	3.786
France	10.241	6.941	5.048
Italy	10.241	5.048	6.31
Cyprus	11.172	8.834	4.417
Latvia	11.979	8.379	8.203
Lithuania	17.303	13.034	8.834
Luxembourg	9.317	4.655	1.262
Hungary	6.941	4.655	3.993
Malta	18.634	13.034	6.31
Netherlands	18.634	11.172	4.417
Austria	8.203	5.586	4.655
Poland	6.655	6.517	5.572
Portugal	10.648	6.941	3.724
Romania	18.634	2.793	1.262
Slovenia	11.979	6.517	3.155
Slovakia	17.303	9.317	6.517
Finland	11.172	6.655	3.155
Sweden	4.655	3.724	1.262
United Kingdom	13.31	9.31	4.417

**Table 1** Computing times for EU countries (in seconds).



**Figure 2** Computing times for EU countries (in seconds).

The values in Table 1 are presented graphically in Figure 2. It is clear that in all cases the calculation module GRG Nonlinear provides solutions in less computing time than the standard module, implemented in MS Excel 2007. In most cases, the Evolutionary module provides a solution before GRG Nonlinear module, but not always. In Figure 3, individual EU countries are organized in descending order of how long it took to refine the parameters of estimates of its Gompertz-Makeham function. The solution provided by individual modules, of course, depends on the input data and on other circumstances. However, we can confirm, that the modules GRG Nonlinear and Evolutionary, implemented in MS Excel 2010, provide a solution sooner than in older versions of MS Excel, what can **bring benefits for specific groups of users and analysts**. It is also important to note that there is probably no relationship between the length of computing time and the size of the considered country. Even if this relationship will be found, this correlation will be certainly misleading. In the case that this study would be performed for another calendar year with other statistics of deaths, the length of computing time would be certainly changed for some countries and the ranking of countries would then also be changed. Probably would not change the order of considered modules.

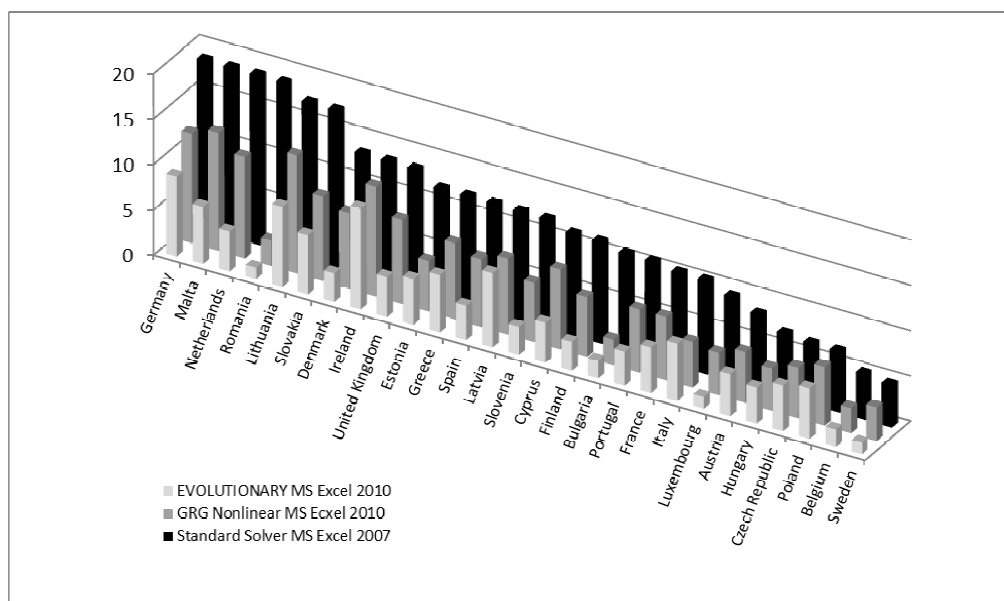


Figure 3 Computing times for EU countries (in seconds, organized in descending order).

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