Hankel max-min matrices and their applications

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Abstract. Hankel matrices are characterized as square matrices with constant entries along every skew-diagonal. Therefore, a Hankel matrix is determined by its entries in the first row and the last column. Special subtype is represented by circulant-Hankel matrices, with entries in every row shifted by one position to the left, in comparison with the previous row. A circulant-Hankel matrix is determined by entries in the first row. Properties of Hankel matrices and circulant-Hankel matrices in max-min algebra are studied in the paper. The question of finding the steady states of complex systems (eigenvectors of the transition matrix) is considered. Specific cases with the maximal matrix input value on different positions are completely described. Applications to real problems, e.g. the influence of viral advertising are presented.

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1 Introduction

Many applications can naturally be solved by using operations maximum and minimum on a linearly ordered set. The so-called max-min algebras have been frequently studied by number of authors and various algorithms have been developed.

Eigenvectors of a max-min matrix characterize stable states of the corresponding discrete-events system. Investigation of the max-min eigenvectors of a given matrix is therefore of great practical importance. The eigenproblem in max-min algebra has been studied by many authors. Interesting results were found in describing the structure of the eigenspace, and algorithms for computing the maximal eigenvector of a given matrix were suggested, see e.g. [1], [2], [4], [8], [9], [10]. The structure of the eigenspace as a union of intervals of increasing eigenvectors was described in [3].

By max-min algebra we understand a triple $(\mathcal{B}, \oplus, \otimes)$, where $\mathcal{B}$ is a linearly ordered set, and $\oplus = \max$, $\otimes = \min$ are binary operations on $\mathcal{B}$. The notation $\mathcal{B}(n, n)$ ($\mathcal{B}(n)$) denotes the set of all square matrices (all vectors) of given dimension n over $\mathcal{B}$. Operations $\oplus$, $\otimes$ are extended to matrices and vectors in a formal way.

The eigenproblem for a given matrix $A \in \mathcal{B}(n, n)$ in max-min algebra consists of finding a vector $x \in \mathcal{B}(n)$ (eigenvector) such that the equation $A \otimes x = x$ holds true. By the eigenspace of a given matrix we mean the set of all its eigenvectors.

In this paper the structure of eigenvectors of Hankel matrices is studied. The advantage of the special matrices is that they enable more efficient solution of some specific problems. Hankel matrices are a special form of matrices, which are determined by the first row and the last column of matrix. The paper presents a detailed description of some possible types of eigenvectors of any given Hankel matrix.

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2 Application

As application of the Hankel matrices we present a simplified model of the information flow in viral marketing. By the source of disseminated information, the messages are divided into groups in which the degree of suggestibility can be monitored. For the sake of simplicity we only work with simplified linear model, which can be described by infinite Hankel matrix. In real application we are working with a finite segment of the infinite linear model, which means its reduction to a finite Hankel matrix.

The object of the model is spreading of information during an advertising campaign in mobile or internet environment, using a mechanism of the viral marketing. A classic environment suitable for application of viral marketing are social networks, such as Facebook, Twitter, Google +, etc.

In the model, transition matrix is considered with entries evaluating the intensity (the importance, etc.) of the links (follow-ups, coherences, etc.) between communication nodes. For any i, j the value a_{ij} is the intensity of the interconnection of the network node i with node j, and a_{ji} is the intensity of the interconnection of node j with node i. The values a_{ij} and a_{ji} must be equal, because they express the intensity of the interconnection, which is the same in both directions when using contemporary mobile technologies. Moreover, the intensity only depends on the source of the disseminated information, thus creating the pairs obtaining the particular information in the same time (with equal communication distance from the source).

Thus, in the linear model we get formal description by Hankel transition matrices which, due to their special form, represent spreading of information with constant intensity in pairs with equal communication distance from the source of information. The transition matrix A transforms the initial state vector x to a new state vector $A \otimes x$, where the state vector entries represent the impact degree of the marketing campaign on the communication nodes.

2.1 Model of suggestibility in viral advertising campaigns

Let us consider n interest groups (nodes in the model) such as they can be found in the social network Facebook. For each group we determine the degree of suggestibility in a viral advertising campaign. The rate can be calculated

$$\text{rate} = \frac{\text{number of advertising messages sent per week in group}}{\text{number of members}} \; .$$

In addition, we determine the level of mutual suggestibility communications between different groups

$$\text{suggestibility level} = \frac{\text{number of advertising messages sent between the two groups per week}}{\text{number of members of both groups}} \; .$$

Furthermore, we set the initial impact degree of the campaigne on each node in the model. The impact degree on each group lies within the interval $\langle 0; 1\rangle$. Level 1 is assigned to a group (node) that is fully affected by the advertising campaign, while the group gets level 0, when the campaign impact on the group is neglectible.

The primal goal of the marketing campaign is to keep the present customers and avoid the unstable behaviour in the market. The stable states of the system correspond to eigenvectors of the transition matrix. For Hankel matrices the eigenproblem can be solved efficiently by considering only the entries in the first row and the last column of the matrix.

By the previous state of the art, it was only known that every constant vector with the value between the minimum and the maximum of the matrix entry values is a stable vector. The disadvantage is that constant eigenvectors can lead either to low penetration in the given marketing segment, or to loss of time and unnecessary costs in addressing the groups that are not advertising enough to hit other groups, or do not use sufficiently many links with other groups. The methods of extremal algebra enable us to describe further eigenvectors, thus helping us to find further (possible all) stable states of the system and increase the efficiency of the viral marketing campaign.

3 Eigenvectors of Hankel matrices

A square matrix is called Hankel matrix is a square matrix with constant skew-diagonals. Hence, Hankel matrix A is fully determined by its inputs in the first row and in the last column.

We shall use the notation $a(n-1) := a_{11}$, $a(n-2) := a_{12}$, ..., $a(1) := a_{1n-1}$, $a(0) := a_{1n}$, $a(-1) := a_{2n}$, ..., $a(2-n) := a_{n-1\,n}$, $a(1-n) := a_{nn}$.

$$\begin{pmatrix} a(n-1) & a(n-2) & \dots & a(1) & a(0) \\ a(n-2) & a(n-3) & \dots & a(0) & a(-1) \\ a(n-3) & a(n-4) & \dots & a(-1) & a(-2) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a(2) & a(1) & \dots & a(4-n) & a(3-n) \\ a(1) & a(0) & \dots & a(3-n) & a(2-n) \\ a(0) & a(-1) & \dots & a(2-n) & a(1-n) \end{pmatrix}$$

We shall use the notation $N^\star = \{1-n, 2-n, \dots, -1, 0, 1, \dots, n-2, n-1\}$. Further we define, for a Hankel matrix A given by the entries $\big(a(k),\, k \in N^\star\big)$ a strictly decreasing sequence $M(A) = (m_1, m_2, \dots)$ of length $s(A)$ by recursion

$$m_t = \begin{cases} \max\{\, a(k);\, k \in N^\star \,\} & \text{for } t = 1 \\ \max\{\, a(k) < m_{t-1};\, k \in N^\star \,\} & \text{for } t > 1 \end{cases}$$

Our research of Hankel matrices is aimed on complete description of all eigenvectors of a given Hankel matrix. The form of the eigenvectors strongly depends on positions of values $m_1, m_2, \dots$ in the entry vector $\big(a(k),\, k \in N^\star\big)$. Three interesting partial cases of our investigation are presented below.

Case 1

In this case we consider a matrix containing the maximum m_1 at one position only, with the remaining matrix entries equal to $m_2 = 0$. All possible eigenvectors are of the form indicated in the table below. In dependence on the position of the maximum entry, the entries of every eigenvector are divided into two groups. In every group the entries can have arbitrary values in the indicated interval, and, with one exception, the values must be symmetric within every entry group.

Position	Symmetry	Value interval	Entry group	Value interval	Entry group
$a(0)$	Yes	$\langle m_1; m_2\rangle$	$\langle x_1; x_n\rangle$	-	-
$a(n-k); k \neq 1, n$	Yes	$\langle m_1; m_2\rangle$	$\langle x_1; x_k\rangle$	$\{0\}$	$\langle x_{k+1}; x_n\rangle$
$a(n-1)$	No	$\langle m_1; m_2\rangle$	x_1	$\{0\}$	$\langle x_2; x_n\rangle$
$a(1-k); k \neq 1, n$	Yes	$\langle m_1; m_2\rangle$	$\langle x_{k+1}; x_n\rangle$	$\{0\}$	$\langle x_1; x_{k-1}\rangle$
$a(1-n)$	No	$\langle m_1; m_2\rangle$	x_n	$\{0\}$	$\langle x_1; x_{n-1}\rangle$

Case 2

In this case we consider a matrix containing the maximum m_1 on two positions, with the remaining matrix entries equal to $m_2 = 0$. The symmetry must now occur simultaneously on two (sometimes only one) entry intervals. If the entry symmetry intervals do not cover the entire vector, the remaining area can only contain zero entries.

Positions	Value interval	Entry symmetry interval
$a(0), a(n-k); k \neq 1, n$	$\langle m_1; m_2 \rangle$	$\langle x_1; x_n \rangle \wedge \langle x_1; x_k \rangle$
$a(0), a(n-1)$	$\langle m_1; m_2 \rangle$	$\langle x_1; x_n \rangle$
$a(n-k), a(n-l); k < l < n$	$\langle m_1; m_2 \rangle$	$\langle x_1; x_k \rangle \wedge \langle x_1; x_l \rangle$
$a(0), a(1-k)$	$\langle m_1; m_2 \rangle$	$\langle x_1; x_n \rangle \wedge \langle x_k; x_n \rangle$
$a(0), a(1-n)$	$\langle m_1; m_2 \rangle$	$\langle x_1; x_n \rangle$
$a(1-k), a(1-l); k < l < n$	$\langle m_1; m_2 \rangle$	$\langle x_k; x_n \rangle \wedge \langle x_l; x_n \rangle$

Case 3

In this case we consider a matrix containing two different maximums m_1, m_2, while the remaining entries are equal to $m_3 = 0$. If the vector field does not cover the entire vector, the remaining area contains only zeros. Vector interval corresponding to the maximum m_1 must be symmetric in values and in positions. If the vector interval corresponding to the second maximum m_2 must be symmetric in positions, but two cases of values greater then m_2 must be considered.

1. Position of m_2 is smaller then position of m_1, then the values on symmetric position should be contained in interval $\langle m_1; m_2 \rangle$.
2. Position of m_2 is greater then position of m_1, then the values on symmetric position must be equal to m_2.

For the inputs on interval $\langle m_2; m_3 \rangle$ the symmetry is requuired on both vector intervals.

4 Comparison with other special matrices

The circulant, Toeplitz, Hankel and circulant-Hankel matrices are often studied as important special matrices in extremal algebras. The matrices have interesting applications, and on the other hand, many algorithms have lower computational complexity in these special cases.

A circulant matrix has a special form given by the entries in the first row. Next rows contain the same values as the first one, but the values are shifted one position to the right. The characteristic digraph of the matrix is symmetric with respect to cyclic permutation of nodes.

Toeplitz matrices are generalization of circulant matrices. A Toeplitz matrix contains constant values on every line parallel with the main diagonal. Hence, the matrix is given by its first row and its first column. Toeplitz matrix can be interpreted as a finite part of an infinite circulant matrix.

Hankel matrices are in some sense dual to Toeplitz matrices, namely they contain constant values on every line parallel with the skew diagonal. Dualy to Toeplitz matrices, all entries in a Hankel matrix are determined by the entries in the first row and in the last column.

Finally, circulant-Hankel matrices represent a special case of Hankel matrices. A circulant-Hankel matrix is given by entries in the first row, and the next rows contain the same values shifted to one position to the left.

There exist also further connection between these special matrix types. In particular, the second power of Hankel matrix is a Toeplitz matrix and third power is again a Hankel matrix. The same connection holds between circulant-Hankel matrices and circulant matrices. As a consequence, an eigenvector of a Hankel matrix is also eigenvector of some Topelitz matrix and an eigenvector of a circulant-Hankel matrix is also eigenvector of some circulant matrix. These connections are illustrated by examples in this section.

Example 1. Let Hankel matrix A with one maximal value on the input $a(0)$ be given.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenvectors corresponding to this matrix are all symmetric vectors containing arbitrary values from interval $\langle 0, 10\rangle$, i.e. vectors of the form $(x_1, x_2, x_3, x_3, x_2, x_1)$.

The second power of Hankel matrix A is a Toeplitz (and in this case also circulant) matrix

$$A^2 = \begin{pmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{pmatrix}$$

Eigenvectors of this matrix are all vectors containing arbitrary values from the interval $\langle 0, 10\rangle$, as it was proved in [5]. Hence, in this example all eigenvectors of the given Hankel matrix A are also eigenvectors of the circulant matrix A^2, but the converse implication does not hold. Some of the eigenvectors of circulant matrix A^2 (the non-symmetric ones) are not eigenvectors of Hankel matrix A.

Example 2. Let Hankel matrix B with maximum on two positions $a(0), a(3)$ be given.

$$B = \begin{pmatrix} 0 & 0 & 10 & 0 & 0 & 10 \\ 0 & 10 & 0 & 0 & 10 & 0 \\ 10 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenspace corresponding to matrix B consists of all vectors symmetric on entry interval $[x_1, x_3]$ and at the same time symmetric on the whole entry interval $[x_1, x_6]$. Under these symetry conditions the eigenvectors can contain arbitrary entry values from interval $\langle 0, 10\rangle$. Hence, every eigenvector of B is of the form $(x_1, x_2, x_1, x_1, x_2, x_1)$ with arbitrary values $x_1, x_2 \in \langle 0, 10\rangle$.

The second power of Hankel matrix B is a Toeplitz (and, similarly as in the previous example, also circulant) matrix

$$B^2 = \begin{pmatrix} 10 & 0 & 0 & 10 & 0 & 0 \\ 0 & 10 & 0 & 0 & 10 & 0 \\ 0 & 0 & 10 & 0 & 0 & 10 \\ 10 & 0 & 0 & 10 & 0 & 0 \\ 0 & 10 & 0 & 0 & 10 & 0 \\ 0 & 0 & 10 & 0 & 0 & 10 \end{pmatrix}$$

Eigenspace of circulant matrix B^2 consists of all eigenvectors containing values form interval $\langle 0, 10\rangle$ which must be repeated every third position, because the greatest common divisor of maximum position and size of matrix is $\gcd(3, 6) = 3$, see [5]. Hence the eigenvectors of B^2 are all vectors of the form $(x_1, x_2, x_3, x_1, x_2, x_3)$ with arbitrary values $x_1, x_2, x_3 \in \langle 0, 10\rangle$.

Similarly as in the previous example, all eigenvectors of the given Hankel matrix B are also eigenvectors of the circulant matrix B^2, but the converse implication does not hold.

Example 3. Let C be a given Hankel matrix with maximum on two positions $a(0), a(2)$.

$$C = \begin{pmatrix} 0 & 0 & 0 & 10 & 0 & 10 \\ 0 & 0 & 10 & 0 & 10 & 0 \\ 0 & 10 & 0 & 10 & 0 & 0 \\ 10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenspace corresponding to matrix C only contains constant vectors with value from interval $\langle 0, 10\rangle$.

Let us remark that in this example the second power of the given matrix is a Toeplitz matrix which in not circulant.

$$C^2 = \begin{pmatrix} 10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 10 & 0 & 0 \\ 10 & 0 & 10 & 0 & 10 & 0 \\ 0 & 10 & 0 & 10 & 0 & 10 \\ 0 & 0 & 10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 & 0 & 10 \end{pmatrix}$$

Eigenspace of Toeplitz matrix C^2 consists of all eigenvectors containing values form interval $\langle 0, 10\rangle$ which must be repeated every second position, because the greatest common divisor of maximum position in the first row and the size of matrix is $\gcd(4, 6) = 2$, see [6]. Hence the eigenvectors of C^2 are all vectors of the form $(x_1, x_2, x_1, x_2, x_1, x_2)$ with arbitrary values $x_1, x_2 \in \langle 0, 10\rangle$.

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