Models of unexpected fluctuations of aggregate income or real interest rate

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Abstract. In this paper, we create two models based on new IS-LM model: the model of unexpected fluctuations of aggregate income and the model of unexpected fluctuations of real interest rate. The new IS-LM model eliminates two main deficiencies of the original model: assumptions of constant price level and of strictly exogenous money supply. The unexpected fluctuations of these quantities can be explained by existence of special type of cycle called relaxation oscillation. Relaxation oscillations include some short part looking like "jumps". These "jumps" can be interpreted like unexpected. The relaxation oscillation is caused by the fiscal or monetary policy. So, these models with relaxation oscillations can be first approximation of the estimation of the government intervention impacts.

Keywords: new IS-LM model, dynamical behaviour, relaxation oscillations, interest rate, aggregate income.

JEL classification: E12
AMS classification: 37N40, 91B50, 91B55

1 Introduction

In this paper, we create the model of unexpected fluctuations of aggregate income and of real interest rate based on new version of IS-LM model which eliminates two main deficiencies of original IS-LM model.

The original IS-LM [4] has two main deficiencies: an assumption of constant price level and of strictly exogenous money supply, i.e. supply of money is certain constant money stock determined by central bank. There exist many versions of IS-LM model and related problems, see e. g. [1, 2, 3, 6, 7, 9, 10], but we created our oun new version of IS-LM model which eliminates these deficiencies.

The model of unexpected fluctuations of aggregate income is based on relaxation oscillations emerging on goods market under certain conditions and the model of unexpected fluctuations of real interest rate is based on relaxation oscillations on money market or financial assets market under certain conditions. Relaxation oscillations include some short part looking like "jumps". These "jumps" can be interpreted like unexpected. The relaxation oscillations emerging on goods or money market are caused by the fiscal or monetary policy. So, this new models with relaxation oscillations can be first approximation of the estimation of the government intervention impacts.

2 Preliminaries

In this section, there are basic notations and a definition of new IS-LM model which eliminates mentioned two main deficiencies of original IS-LM model.

2.1 Basic Notations

Y	aggregate income (GDP, GNP)	I	investments	L	demand for money
R	long-term real interest rate	S	savings	M	supply of money
i_S	short-term nominal interest rate				

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2.2 New IS-LM model

We assume a two-sector economy, a demand-oriented model, $Y \geq 0,\ R \in \mathbb{R},\ i_S \in \mathbb{R}^+$, a variable price level, a conjunction of endogenous and exogenous money supply.

So, we add to original IS-LM model a floating price level, i.e. inflation effect. So, we need to distinguish two type of interest rate. There is the long-term real interest rate on goods market and the short-term nominal interest rate on money (or financial assets) market. The well-known relation holds:

$$i_S = R - MP + \pi^e, \tag{1}$$

where MP is a maturity premium and π^e is an inflation rate. While MP and π^e are constants, it holds $\frac{di_S}{dt} = \frac{d(R-MP+\pi^e)}{dt} = \frac{dR}{dt}$.

Then, we consider that the money supply is not strictly exogenous quantity, but the supply of money is endogenous quantity (money are generated in economics by credit creation, see e.g. [8]) with some exogenous part (certain money stock determined by Central bank).

Definition 2.1. We define the *supply of money* by the formula

$$M(Y, i_S) + M_S, \tag{2}$$

where $M(Y, i_S)$ represents the endogenous part of money supply and $M_S > 0$ represents the exogenous part of money supply.

So, there are the investment function $I(Y, R)$ and the saving function $S(Y, R)$ on goods market and the demand for money function $L(Y, i_S) = L(Y, R - MP + \pi^e)$ and money supply function $M(Y, i_S) = M(Y, R - MP + \pi^e)$ on the money market. It also holds $\frac{\partial L(Y,i_S)}{\partial i_S} = \frac{\partial L(Y,R-MP+\pi^e)}{\partial R}$ and $\frac{\partial M(Y,i_S)}{\partial i_S} = \frac{\partial M(Y,R-MP+\pi^e)}{\partial R}$ because of constant MP and π^e. We suppose that all of these functions are continuous and differentiable.

Definition 2.2. The *new IS-LM model* is given by the following system of two algebraic equations

$$\begin{aligned} &\text{IS:} \quad I(Y, R) = S(Y, R) \\ &\text{LM:} \quad L(Y, R - MP + \pi^e) = M(Y, R - MP + \pi^e) + M_S, \end{aligned} \tag{3}$$

where $M_S > 0$, in the static form and by this system of two differential equations

$$\begin{aligned} &\text{IS:} \quad \frac{dY}{dt} = \alpha[I(Y, R) - S(Y, R)] \\ &\text{LM:} \quad \frac{dR}{dt} = \beta[L(Y, R - MP + \pi^e) - M(Y, R - MP + \pi^e) - M_S], \end{aligned} \tag{4}$$

where $\alpha, \beta > 0$, in the dynamic form.

Economic theory puts on the main economic functions of this model some properties. We can present these properties using the following formulas:

$$0 < \frac{\partial I}{\partial Y} < 1, \frac{\partial I}{\partial R} < 0, 0 < \frac{\partial S}{\partial Y} < 1, \frac{\partial S}{\partial R} > 0, \tag{5}$$

$$\frac{\partial L}{\partial Y} > 0, \frac{\partial L}{\partial R} < 0. \tag{6}$$

We have to put on our new function of money supply some economic properties. These properties are

$$0 < \frac{\partial M}{\partial Y} < \frac{\partial L}{\partial Y}, \frac{\partial M}{\partial R} > 0. \tag{7}$$

The first formula means that the relation between supply of money and aggregate income is positive and that the rate of increase of money supply depending on aggregate income is smaller than the rate of increase of money demand depending on aggregate income because the banks are more cautious than another subjects. And the second formula means that the relation between supply of money and interest rate is positive.

It is easy to see (using the Implicit Function Theorem) that the course of the curve LM is increasing in this new IS-LM model with properties (6) and (7). If the inequality

$$\frac{\partial I}{\partial Y} < \frac{\partial S}{\partial Y}, \tag{8}$$

also holds in additions to the properties (5), then the course of the curve IS will be decreasing. Now, we denote the function, whose graph is the curve IS, as $R_{IS}(Y)$ and the function, whose graph is the curve LM, as $R_{LM}(Y)$. These functions exist because of the Implicit Function Theorem. Now, if we assume

$$\lim_{Y \to 0^+} R_{IS}(Y) > \lim_{Y \to 0^+} R_{LM}(Y), \tag{9}$$

then there will exist at least one intersection point of the curve IS and LM.

3 Model of unexpected fluctuations of aggregate income (relaxation oscillations on goods market)

In this case, we assume that the subjects and their reactions on goods market are faster than on money market. So, we suppose that the interest rate R is changing very slowly in time in proportion to the aggregate income Y. We can describe this situation by following equations

$$\begin{aligned} \frac{dY}{dt} &= \alpha[I(Y,R) - S(Y,R)] \\ \frac{dR}{dt} &= \epsilon\beta[L(Y, R - MP + \pi^e) - M(Y, R - MP + \pi^e) - M_S] \end{aligned} \tag{10}$$

where $\epsilon \to 0$ is some very small positive parameter.

If this parameter ϵ is very small, then we can consider $\frac{dR}{dt} = 0$ and we can write previous equations in the following forms.

$$\begin{aligned} \frac{dY}{dt} &= \alpha[I(Y,R) - S(Y,R)] \\ \frac{dR}{dt} &= 0 \end{aligned} \tag{11}$$

Now, we formulate sufficient conditions for existence of relaxation oscillations on goods market. We tend to Kaldor's theory about investment and saving function, see [5]. The investment and saving function depends only on Y for some fixed R ($I(Y)$ and $S(Y)$) has so-called "sigma-shaped" graphs, see Figure 1.

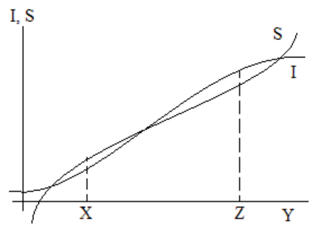

Figure 1: Shapes of the graphs of $I(Y)$ and $S(Y)$

Now, we use (with another designations) the description of these properties from Baráková, see [1]. So-called *"Kaldor's" conditions* are

$$\begin{aligned} \frac{\partial I}{\partial Y} &< \frac{\partial S}{\partial Y} \quad \text{for} \quad Y \in [0, X), \\ \frac{\partial I}{\partial Y} &> \frac{\partial S}{\partial Y} \quad \text{for} \quad Y \in (X, Z), \\ \frac{\partial I}{\partial Y} &< \frac{\partial S}{\partial Y} \quad \text{for} \quad Y \in (Z, \infty), \end{aligned} \tag{12}$$

where points $X < Z$ are given by equation $\frac{\partial I}{\partial Y} = \frac{\partial S}{\partial Y}$ for some fixed R.

Now, we consider the system (10) with requirement of $\epsilon \to 0$ and with properties (5), (6), (7) and (12) (but we can reconsider the system (10) in the simplified way (11)). There we can see that the variable R is a parameter in equations $\frac{dY}{dt} = \alpha[I(Y,R) - S(Y,R)]$. There, every points of the curve IS are singular points. On the next Figure 2 we can see some displaying of relaxation oscillations on goods market.

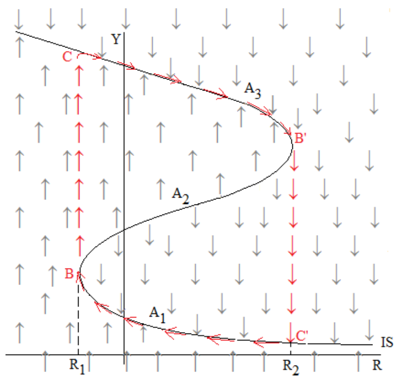

Figure 2: Relaxation oscillations on goods market

The trajectories of this system are directed almost vertically downwards or upwards (parallel to axis Y) considering $\frac{dR}{dt} = 0$. Up or down direction of the trajectories is given by the sign of the function $\beta[L(Y, R-MP+\pi^e) - M(Y, R-MP+\pi^e) - M_S]$ on the curve IS. So, the direction of these trajectories is dependent on the stability or unstability of the arcs A_1, A_2 and A_3. The arcs A_1 and A_3 are stable arcs and the arc A_2 is unstable arc, see Figure 2. It follows from qualification of possible singular points in the system (10) excluding $\epsilon \to 0$. The trajectories are attracted to the stable arcs A_1 or A_3 and are drove away the unstable arc A_2. The velocity of trajectories are finite near the isocline IS and nearness of the curve IS the trajectories go along the curve IS. The velocity of trajectories are infinite large elsewhere. Now, we construct the cycle which is one vibration of the relaxation oscillations. We are changing the parameter R from the level R_2 to R_1. If the moving point is on or near the stable arc A_1, the moving point will go along this stable arc A_1, then it will pass the unstable arc A_2 from point B to C, see Figure 2. The velocity between the point B and C is infinite large. There originates some "jump". There is the similar situation if we are changing the parameter R from the level R_1 to R_2. This oscillation contains the trajectories described by stable arcs A_1 and A_3 with finite velocity and the trajectories described by vertical segments (between points B and C and also between B' and C') with infinite velocity (looking like a "jump"). These trajectories form clockwise cycle.

These changes of the variable R can be caused by the monetary policy and then there originates describing quick "jump". This jump seems to be unexpected.

4 Model of unexpected fluctuations of real interest rate (relaxation oscillations on money market, or financial assets market)

In this case, we assume that the subjects and their reactions on money market are faster than on goods market. So, we suppose that aggregate income Y is changing very slowly in time in proportion to the interest rate R. We can describe this situation by following equations

$$\begin{aligned} \frac{dY}{dt} &= \epsilon\alpha[I(Y,R) - S(Y,R)] \\ \frac{dR}{dt} &= \beta[L(Y, R-MP+\pi^e) - M(Y, R-MP+\pi^e) - M_S] \end{aligned} \tag{13}$$

where $\epsilon \to 0$ is some very small positive parameter.

If this parameter ϵ is very small, then we can consider $\frac{dY}{dt} = 0$ and we can write previous equations in the following forms.

$$\begin{aligned} \frac{dY}{dt} &= 0 \\ \frac{dR}{dt} &= \beta[L(Y, R-MP+\pi^e) - M(Y, R-MP+\pi^e) - M_S] \end{aligned} \tag{14}$$

Now, we formulate sufficient conditions for existence of relaxation oscillations on money or financial assets market. We suppose some unusual behaviour of the demand for money and supply of money. We assume so-called *three phases money demand and money supply* depending on i_S for some fixed Y. In the first phase, for $i_S \in [0, P), P > 0$, these subjects on the money (or financial assets) market behave usual and the course of the money demand and money supply function is standard how we describe in the condition (6) and (7). In the second phase, for $i_S \in (P,Q), P < Q$, these subjects behave unusual, precisely reversely. We can describe this behaviour using following formula

$$\frac{\partial L}{\partial i_S} = \frac{\partial L}{\partial R} > 0, \frac{\partial M}{\partial i_S} = \frac{\partial M}{\partial R} < 0. \tag{15}$$

These properties correspond to unusual economic situation called liquidity trap. This means that the subjects on money or financial assets market (demand side) prefer liquidity despite relatively high level of (short-term nominal) interest rate. They own money rather than stocks, although they could have bigger gain because of relative high level of (short-term nominal) interest rate. This "irrational" behaviour of these subjects can be caused by big risk of holding these stocks and by small willingness to undergo this risk. The supply of money fully adapts to money demand (we assume demand-oriented model). This phase should be small. In the third phase, for $i_S \in (Q, \infty)$, these subjects behave usual as in the first phase. We can see the graphs of money demand and money supply function describing this behaviour on the following Figure 3.

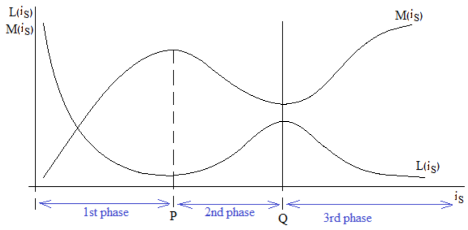

Figure 3: Three phases of the graphs of $L(i_S)$ and $M(i_S)$

Remark 4.1. $\frac{\partial L}{\partial i_S} = \frac{\partial L}{\partial R} = 0$ and $\frac{\partial M}{\partial i_S} = \frac{\partial M}{\partial R} = 0$ in the point P and Q.

Now, we consider the system (13) with requirement of $\epsilon \to 0$ and with properties (5), (8) and three phases money demand and money supply (but we can reconsider the system (13) in the simplified way (14)). There we can see that the variable Y is a parameter in equations $\frac{dR}{dt} = \beta[L(Y, R - MP + \pi^e) - M(Y, R - MP + \pi^e) - M_S]$. There, every points of the curve LM are singular points. On the next Figure 4 we can see some displaying of relaxation oscillations on money or financial assets market.

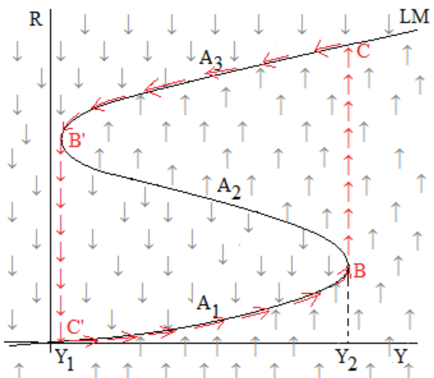

Figure 4: Relaxation oscillations on money (or financial assets) market

The trajectories of this system are directed almost vertically downwards or upwards (parallel to axis R) considering $\frac{dY}{dt} = 0$. Up or down direction of the trajectories is given by the sign of the function $\alpha[I(Y,R) - S(Y,R)]$ on the curve LM. So, the direction of these trajectories is dependent on the stability or unstability of the arcs A_1, A_2 and A_3. The arcs A_1 and A_3 are stable arcs and the arc A_2 is unstable arc, see Figure 4. It follows from qualification of possible singular points in the system (13) excluding $\epsilon \to 0$. The trajectories are attracted to the stable arcs A_1 or A_3 and are drove away the unstable arc A_2. The velocity of trajectories are finite near the isocline LM and nearness of the curve LM the trajectories go along the curve LM. The velocity of trajectories are infinite large elsewhere. Now, we construct the cycle which is one vibration of the relaxation oscillations. We are changing the parameter Y from the level Y_1 to Y_2. If the moving point is on or near the stable arc A_1, the moving point will go along this stable arc A_1, then it will pass the unstable arc A_2 from point B to C, see Figure 2. The velocity between the point B and C is infinite large. There originates some "jump". There is the similar situation if we are changing the parameter Y from the level Y_2 to Y_1. This oscillation contains the trajectories described by stable arcs A_1 and A_3 with finite velocity and the trajectories described by vertical segments (between points B and C and also between B' and C') with infinite velocity (looking like a "jump"). These trajectories form counterclockwise cycle.

These changes of the variable Y can be caused by the fiscal policy and then there originates describing quick "jump". This jump seems to be unexpected.

Conclusion

In these days, many experts and also the public more and more talk about unexpected fluctuation of different phenomenons in economics and about impact of these fluctuations on economics. We try to model some unexpected fluctuation of aggregate income on goods market and of interest rate on money or financial assets market using own new IS-LM model and theory of relaxation oscillations. New IS-LM model differs from the original model in elimination of its two main deficiencies (assumption of constant price level and of strictly exogenous money supply).

Relaxation oscillations on goods market cause quick change of the aggregate income which seems to be unexpected. Similarly, relaxation oscillations on money market (or financial assets market) cause quick change of the interest rate which seems to be unexpected. This quick "jumps" can ascribe unusual behaviour of economics. Relaxation oscillations on money or financial assets market can be caused by fiscal policy and on goods market by monetary policy. So, this new IS-LM model with relaxation oscillations can be first approximation of the estimation of the government intervention impacts.

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