Application of methodology Value at Risk for market risk with normal mixture distribution

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Abstract. Conditional Value at Risk (CVaR) has been proposed as an alternative, almost coherent, risk measure to Value at Risk (VaR), as it considers expected loss beyond VaR. This paper deals with estimating Value at Risk and conditional Value at Risk under the assumption of mixture normal distribution. We apply mixture normal distribution by assuming that the economy is in various phases of the business cycle. We determinate the both risk measures for market risk, daily returns of popular indices (DAX, CAC, Nikkei and FTSE) over ten years. In the first part, we describe methodology VaR and CVaR and techniques of estimating parameters of probability distributions are presented, i.e. general method of moments and maximum likelihood. Finally, we compare all estimates with each other.

Keywords: mixture distribution, EM algorithm, method of moments, Value at Risk.

JEL Classification: C16, G22, G32
AMS Classification: 91B30

1 Introduction

The Value at Risk (VaR) is defined as the maximum potential loss in value of a portfolio due to adverse market movements, for a given probability. There is also a possibility to refer to VaR as managing risk methodology which is applied widely to modeling credit, operational, market and also insurance risk. Value at Risk is very easy concept; its measurement is a very challenging statistical problem. A good introduction to Value at Risk methodology is provided by the technical document from [11] or by many follow-up books such as [2], [8], [9], [10].

Nevertheless, we can find a lot of VaR criticism, for instance [10]. [3] deals with the features of good risk measure (called coherent) which is defined by four assumption imposed on the ideal risk measure, i.e. monotonicity, sub-additivity, homogeneity and translational invariance. Value at Risk satisfies all these features only in specific case. Specifically, the sub-additivity is violated as far as the portfolio´s profit/loss or portfolio´s return cannot be characterized by some elliptical probability distribution; see [4] for more details. In addition, the VaR says nothing about the loss behind the VaR. Therefore, other risk measures are preferable such as conditional Value at Risk (CVaR) which represents the average of losses exceeding the VaR.

Moreover, if the VaR and CVaR are estimated analytically, the distribution assumption is needed. Normal distribution can be supposed but this assumption results in underestimation of VaR and CVaR due to the existence of fat tails. In facts, to solve this problem, only two approaches seem to be applicable. One can consider Extreme Value Theory (EVT) focused on fitting the tail distribution only which is approximated mostly via general Pareto distribution, the other can apply mixture distribution to fit the empirical distribution the most. The series of studies are devoted the heavy tails [7], [12].

Thus, the aim of paper is estimating Value at Risk and conditional Value at Risk under the assumption of mixture normal distribution and normal distribution. To respect the fat tails, we apply normal mixture distribution. We highlight in this paper that the VaR and CVaR are highly underestimated in that case.

The paper is organized as follows. Section 2 is devoted to the description of normal mixture distribution and methods of its estimate parameters and general Value at Risk methodology under assumption normal mixture distribution. The VaR and CVaR estimates under normal distribution and mixture probability distribution are determined in Section 3 and Section 4 concludes the paper.

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2 Value at Risk methodology

In this part, we focus normal mixture distribution and then we describe methods of estimating the parameters of the normal mixture distribution, i.e. maximum likelihood estimation (MLE) and method of moments (MM). Finally, we characterize shortly Value at Risk and Conditional Value at Risk.

2.1 Normal mixture distribution

The financial analysis is based generally on the assumption of normally distributed returns but we know that this assumption is not valid. In most financial markets returns are both skewed and leptokurtic. Hence, a number of alternatives skewed and leptokurtic distributions have been applied. Suitable alternative to normal distribution is a mixture of two or more normal distributions. Normal mixture distribution is described in [6].

We can define the mixture distribution like a probability-weighted sum of other distribution functions. For instance, in a mixture of two normal distributions, there are two regimes for returns: one where the return has mean μ_1 and variance σ_1^2 and another where the return has mean μ_2 and variance σ_2^2. The parameter π is the probability for first regime, so the second regime occurs with probability 1- π. The mixture distribution of just two normal densities is defined by

$$G(x) = \pi F(x:\mu_1,\sigma_1^2) + (1-\pi)F(x:\mu_2,\sigma_2^2), \qquad 0 \prec \pi \prec 1, \tag{1}$$

where $F(x:\mu_i,\sigma_i^2)$ denotes the normal distribution function with mean μ_i and variance σ_i^2, for i = 1,2, and where π is the probability associated with the normal component with mean μ_1 and variance σ_1^2. We suppose that the expectation of sample is zero for both distribution, so $\mu_1 = \mu_2$ = 0. In this case the variance of the normal mixture distribution is just the probability- weighted sum of distribution functions, i.e.

$$\sigma^2 = \pi\sigma_1^2 + (1-\pi)\sigma_2^2. \tag{2}$$

The skewness is zero and the kurtosis is given by

$$\kappa = 3\left(\frac{\pi\sigma_1^4 + (1-\pi)\sigma_2^4}{\left[\pi\sigma_1^2 + (1-\pi)\sigma_2^2\right]^2}\right). \tag{3}$$

2.2 Mixture parameter estimation

The estimation of the mixture parameter can be via two methods: maximum likelihood method and method of moments.

Maximum likelihood estimation is a general method for estimating the parameters of a distribution. This method is used extensively because maximum likelihood estimators are consistent. That is, the distribution of the estimator converges to the true value of the parameter as the sample size increases. For estimating the parameter of mixture distribution via method maximum likelihood is used the EM algorithm. The Expectation maximization (EM) algorithm is an efficient iterative procedure to compute the Maximum Likelihood (ML) estimate in the presence of missing or hidden data. Each iteration of the EM algorithm consists of two processes: The E-step, and the M-step. The E-step is the calculation of the expected log likelihood given the current estimates of and given some distribution on the values of the latent variable. In the M-step, the likelihood function is maximized under the assumption that the missing data are known. EM algorithm is described in [1].

As the number of distributions in the mixture increases the probability weight on some of these components can become extremely small. However, in finance it is seldom necessary to use more than two or three components in the mixture, since financial asset return distributions are seldom so irregular as to have multiple modes. In this approach we equate the first few sample moments (one moment for each parameter to be estimated) to the corresponding theoretical moments of the normal mixture distribution. *The method of moments* in general provides estimators which are consistent but not as efficient as the maximum likelihood ones. If we will be apply method of moments for estimate the parameters of a normal mixture distribution we equate to the first four sample moments by changing the parameters of the normal mixture distribution. The vector of probability weights is denoted by $\pi = (\pi_1, \ldots\ldots, \pi_m)$ where $\sum_{i=1}^{m} \pi_i = 1$. The non-central moments are

$$\begin{aligned}
M_1 &= E[X] = \sum_{i=1}^{m} \pi_i \mu_i \\
M_2 &= E[X^2] = \sum_{i=1}^{m} \pi_i (\sigma_i^2 \mu_i^2) \\
M_3 &= E[X^3] = \sum_{i=1}^{m} \pi_i (3\mu_i \sigma_i^2 + \mu_i^3) \\
M_4 &= E[X^4] = \sum_{i=1}^{m} \pi_i (3\sigma_i^4 + 6\mu_i^2 \sigma_i^2 + \mu_i^4).
\end{aligned} \tag{4}$$

And the mean (μ), variance (σ^2), skewness (τ) and kurtosis (κ) are

$$\begin{aligned}
\mu &= E[X] = M_1 \\
\sigma^2 &= E[(X-\mu)^2] = M_2 - M_1^2 \\
\tau &= \sigma^{-3} E[(X-\mu)^3] = \sigma^{-3}(M_3 - 3M_1 M_2 + 2M_1^3) \\
\kappa &= \sigma^{-4} E[(X-\mu)^4] = \sigma^{-4}(M_4 - 4M_1 M_3 + 6M_1^3 M_2 + 2M_1^4).
\end{aligned} \tag{5}$$

The characteristics of particular moments are described in various books for example [1].

2.3 Value at Risk and Conditional Value at Risk

Value at Risk is defined as the smallest loss for the predicted level of probability for a given time interval. It is a function of two parameters, i.e. the risk horizon and the confidence level. We can also characterize the Value at Risk as a one-sided confidence interval of potential loss of portfolio value for a given holding period, which can be written:

$$F(x) = P\left(X \le -VaR_{\alpha,\Delta t}(x)\right) = \alpha\,, \tag{6}$$

where $F(x)$ is cumulative distribution function, α is significance level and Δt is holding period or risk horizon.

For normal distribution we can write VaR estimation as

$$VaR = \Phi^{-1}(1-\alpha)\sigma - \mu \tag{7}$$

where Φ is standard normal distribution function, μ expectation and σ standard deviation. We can determinate VaR from the mixture distribution function thus

$$G(x) = \pi P\left(X \prec (x_a - \mu_1)\sigma_1^{-1}\right) + (1-\pi)\left(X \prec (x_a - \mu_2)\sigma_2^{-1}\right) = \alpha. \tag{8}$$

X is normal variable, we know its quartiles. That is, we know everything in the above identity except the mixture quantile x_α. We can find the mixture quantile using an iterative approximation method such as the goal programming and VaR for mixture distribution is $VaR_\alpha = -x_\alpha$.

Conditional Value at Risk informs what the losses would exceed this level. Conditional VaR can be generally defined in the form of

$$CVaR_{\alpha,\Delta t} = -\alpha^{-1} \int_{-\infty}^{VaR_{\alpha,\Delta t}} x f(x)\, dx, \tag{9}$$

where $f(x)$ is density function. CVaR is computed by assuming normal distribution as

$$CVaR_\alpha = \alpha^{-1} \varphi\left(\Phi^{-1}(\alpha)\right)\sigma - \mu \tag{10}$$

where φ and Φ is standard normal density and distribution function, μ expectation and σ standard deviation. We can write formula for CVaR by assuming the mixture distribution as

$$CVaR_\alpha = \alpha^{-1} \sum_{i=1}^{n} \left(\pi_i \sigma_i \varphi\left(\sigma_i^{-1} x_\alpha\right)\right) - \sum_{i=1}^{n} \pi_i \mu_i. \tag{11}$$

3 Determination Value at Risk for mixture distributions

We will apply Value at Risk for four equities indices (CAC 40, DAX, FTSE 100 and NIKKEI 25). We will estimate VaR and CVaR at 99.9% and 99.5% confidence level over one day risk horizon, because the financial institutions determinate capital requirement for market risk for confidence level (banks at 99.9% and insurance companies at 99.5% confidence level.) We prefer one day risk horizon because it is different in financial institutions (for banks is 10 days and insurance companies 1 year horizon). The period sample is between January 2002 and December 2011 and we take the daily log returns for the stock indices over the whole period. The basic numerical characteristics of individual stock returns, especially the mean, standard deviation, kurtosis, skewness are shown in the following table. Histograms of empirical values you can see in Figure 1 – 4.

Variable	Mean	Volatility	Skewness	Kurtosis
CAC 40	-0.0145%	1.5946%	0.0867	5.2427
DAX	0.0044%	1.6478%	0.0678	4.4742
FTSE 100	0.0026%	1.3357%	-0.1195	6.4006
NIKKEI	-0.0076%	1.5748%	-0.4764	7.7677

Table 1 Moments of equities indices

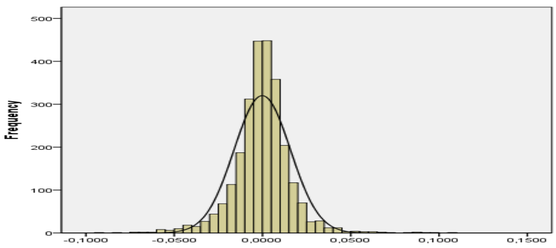

Figure 1 Log returns of CAC

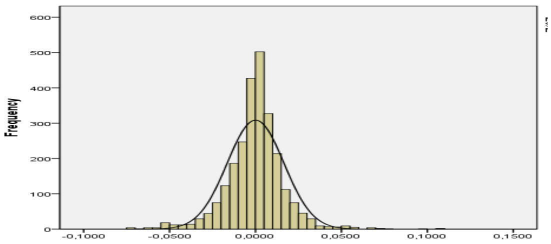

Figure 2 Log returns of DAX

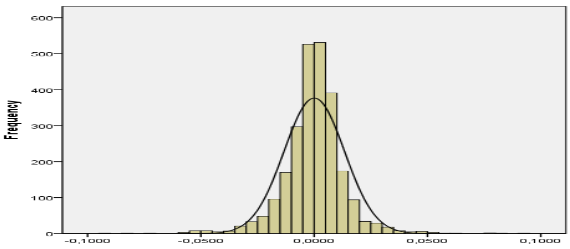

Figure 3 Log returns of FTSE

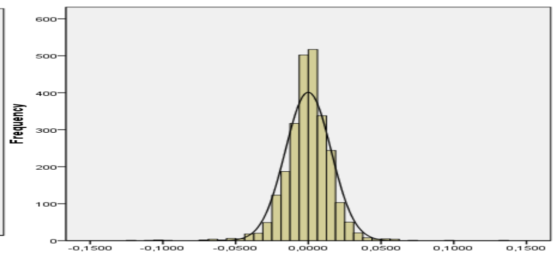

Figure 4 Log returns of NIKKEI

Firstly, we estimate parameters of normal mixture distributions. We apply the method of moment and EM algorithm to fit a mixture of two normal distributions to the daily returns for the four equities indices (CAC 40, DAX, FTSE 100 and NIKKEI 25). We can see estimated parameters in Table 2.

	Method of moments				Maximum likelihood			
	CAC 40	**DAX**	**FTSE 100**	**NIKKEI**	**CAC 40**	**DAX**	**FTSE 100**	**NIKKEI**
μ_1	0.026%	0.342%	-0.214%	0.007%	-0.045%	0.038%	0.064%	0.081%
μ_2	0.065%	-0.239%	0.082%	0.059%	0.494%	-0.074%	0.052%	0.047%
σ_1	2.085%	1.672%	1.559%	1.020%	1.850%	1.907%	1.559%	1.255%
σ_2	0.849%	0.897%	0.188%	2.256%	0.870%	0.830%	0.588%	0.867%
π_1	0.342	0.592	0.723	0.658	0.650	0.230	0.810	0.380
π_2	0.658	0.408	0.277	0.342	0.350	0.770	0.190	0.620

Table 2 Estimated parameters of normal mixture distribution

We can see differences for various methods of estimate parameters. The highest differences are in parameters π_1. Finally, VaR and CVaR were determined under assumption normal mixture distribution and normal distribution. We use significance levels α = 0.5% and 0.1% and time horizon one day. For each of the four risk factor we use Solver to back out the normal mixture VaR from the equation (8). We calculate VaR model with estimate parameters via method of maximum likelihood and method of moments. The results are in the next tables.

VaR model	Normal Mixture		Normal	Normal Mixture		Normal
	MM	MLE		MM	MLE	
Significance	0.1%			0.5%		
CAC 40	7.76%	8.02%	4.91%	4.58%	4.71%	4.09%
DAX	8.06%	8.32%	5.10%	4.76%	4.64%	4.25%
FTSE 100	6.53%	6.27%	4.13%	3.86%	3.98%	3.44%
NIKKEI	7.68%	7.94%	4.86%	4.53%	4.66%	4.05%

Table 3 The results VaR model

Value VaR under assumption normal distribution is much significantly lower than VaR under assumption normal mixture distribution. The value of VaR is so undervalued, which also leads held by the low economic capital to cover potential risks.

CVaR model	Normal Mixture		Normal	Normal Mixture		Normal
	MM	MLE		MM	MLE	
Significance	0.1%			0.5%		
CAC 40	6.97%	7.22%	5.39%	6.37%	6.49%	5.68%
DAX	7.13%	8.71%	5.55%	6.56%	6.68%	5.85%
FTSE 100	6.08%	6.08%	4.50%	5.32%	5.44%	4.75%
NIKKEI	6.89%	6.89%	5.31%	6.28%	6.40%	5.61%

Table 4 The results CVaR model

The same results we conclude according to the CVaR results. Also in this case, it is obvious that the CVaR estimates are highly underestimated. Thus, the importance of applying normal mixture distribution to quantify the risk measure in the form of VaR or CVaR is obvious and we can highly recommend it.

4 Conclusion

The paper deals with quantification of risk measure using Value at Risk methodology on market risk for four equities indices (CAC 40, DAX, FTSE 100 and NIKKEI 25). Firstly, we characterize normal mixture distribution and determination VaR and CVaR. Subsequently, we estimated parameters mixture distribution via the maximum likelihood method and method of moments. Finally, we determined VaR and CVaR at 99.9% and 99.5% confidence level over one day risk horizon and we compared estimates of both risk measures on the assumption that normal distribution with estimates under assumption normal mixture distribution.

We know that asset returns tend to be skewed and heavy tails. Only just normal mixture distribution can model heavy tails and we supposed the fat tail of probability distribution and therefore we applied normal mixture distribution. Thus, we do not take fat tails into account it can lead to the very imprecise and very different Value at Risk estimates resulting in insufficient capital which should cover the loss.

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