Cumulative prospect theory and almost stochastic dominance in valuation of decision alternatives

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Abstract. Searching for good decision rules is one of the most important direction of research in decision making. There are well known and objective rules consistent with rationality, for example stochastic dominance rules, but, according to research based on behavioural approach, decision makers don't always act rationally. Relatively new tools, which model real choices, are cumulative prospect theory rules and almost stochastic dominance rules. The aim of our paper is to examine the consistency of the valuation of decision alternatives based on the cumulative prospect theory and the almost stochastic dominance rules. We show that choices made on the basis of considered tools are not always consistent, but the identification of causes of this discrepancy needs further research.

Keywords: cumulative prospect theory, almost stochastic dominance, stochastic dominance, decision making.

JEL Classification: D81
AMS Classification: 91B06

1 Introduction

For years the most researchers of the decision theory try to find new tools which will better model real decision-makers' choices. The prospect theory based on behavioral economics was one of such tools. Authors of the prospect theory were accused of its inconsistency with the stochastic dominance rules. That problem was solved by the cumulative prospect theory [13]. On the other hand also the stochastic dominance rules do not settle decision situations which would appear obvious. It forced some relaxation of those rules in the form of the almost stochastic dominance rules [6].

The aim of our paper is to examine the consistency of valuation of decision alternatives based on cumulative prospect theory and almost stochastic dominance rules.

2 Cumulative prospect theory

The prospect theory is one of the decision theories which try to explain the way decision-makers make their decisions in the situations of risk. In cumulative prospect theory [5, 13] the phase of evaluation of random decision alternative (prospect) is preceded by the editing phase. The aim of the editing phase is to organize and reformulate the prospects. Possible outcomes of prospect are transformed into gains and losses relative to some reference point which can represent the desirable or actual level of wealth. Then the representation of random decision alternative is different from that in the expected utility theory, in which the absolute levels of wealth are considered. Moreover, in the editing phase probabilities associated with the same outcomes are aggregated what simplifies further evaluation. As a result of the editing phase we obtain the prospect L represented as a sequence of relative outcomes x_i and corresponding probabilities p_i

$$\mathrm{L} = \big((x_1, p_1);\ldots;(x_k, p_k);(x_{k+1}, p_{k+1});\ldots;(x_n, p_n)\big)$$

where $x_1 < \ldots < x_k < 0 \le x_{k+1} < \ldots < x_n$ and $p_1 + \ldots + p_k + p_{k+1} \ldots + p_n = 1$.

In the second phase (evaluation phase) the value of each prospect is calculated. This value depends on two functions: value function $v(x)$ and probability weighting function $g(p)$. The analytical form of the value function and the evaluation of its parameters are determined on the basis of revealed preferences of decision-makers. In the literature various examples of the value function can be found (Dudzińska-Baryła and Kopańska-Bródka proposed quadratic function [2]) but the most cited is

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$$v(x) = \begin{cases} -\lambda(-x)^{\beta}, & x < 0 \\ x^{\alpha}, & x \geq 0 \end{cases}$$

in which values of parameters α, β and λ are 0.88, 0.88 and 2.25 respectively [13].

The plot of the value function is showed in figure 1. The S-shape of the value function is justified by the fact that losses make decision-maker risk-prone but when faced gains decision-maker is risk-averse.

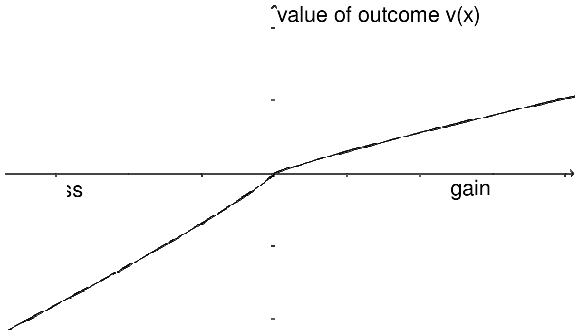

Figure 1 The value function

In cumulative prospect theory there is also considered the fact that decision-makers do not follow the objective probabilities. Therefore Tversky and Kahneman have proposed non-linear transformation of probabilities in the following probability weighting function

$$g(p) = \frac{p^{\gamma}}{\left[p^{\gamma} + (1-p)^{\gamma}\right]^{1/\gamma}}$$

where the value of parameter γ depends on that whether probability concerns gains or losses. In the literature [13] γ is 0.61 for gains and γ is 0.69 for losses. The plot of the probability weighting function is displayed in figure 2.

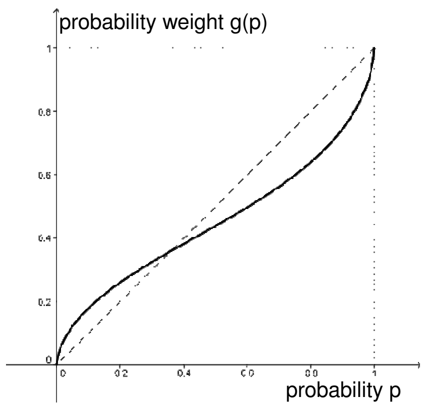

Figure 2 The probability weighting function

The probability weighting function has to have following features: it is increasing function, it overweights low probabilities, and underweights moderate and high probabilities, moreover $g(0) = 0$, $g(1) = 1$ and $g(p) + g(1-p) < 1$ for all $p \in (0,1)$. In the literature various probability weighting functions are analyzed, e.g. Currim and Sarin [1] have proposed four different forms of $g(p)$, and Prelec [12] and Wu and Gonzalez [14, 3] examined features of these functions.

Based on $v(x)$ and $g(p)$ functions the measure of the prospect value is constructed, which is the sum of the evaluation of gains $\mathrm{CPT}^{+}(\mathbf{x},\mathbf{p})$ and the evaluation of losses $\mathrm{CPT}^{-}(\mathbf{x},\mathbf{p})$ [13]:

$$\mathrm{CPT}(\mathbf{x},\mathbf{p}) = \mathrm{CPT}^{+}(\mathbf{x},\mathbf{p}) + \mathrm{CPT}^{-}(\mathbf{x},\mathbf{p})$$

Components $\mathrm{CPT}^{+}(\mathbf{x},\mathbf{p})$ and $\mathrm{CPT}^{-}(\mathbf{x},\mathbf{p})$ are calculated as follows:

$$\mathrm{CPT}^{+}(\mathbf{x},\mathbf{p}) = v(x_n)g(p_n) + \sum_{i=k+1}^{n-1} v(x_i)\left[g\left(\sum_{j=i}^{n} p_j\right) - g\left(\sum_{j=i+1}^{n} p_j\right)\right]$$

$$\mathrm{CPT}^{-}(\mathbf{x},\mathbf{p}) = v(x_1)g(p_1) + \sum_{i=2}^{k} v(x_i)\left[g\left(\sum_{j=1}^{i} p_j\right) - g\left(\sum_{j=1}^{i-1} p_j\right)\right]$$

In the evaluation phase for each prospect the measure $\mathrm{CPT}(\mathbf{x},\mathbf{p})$ is calculated. Among all prospects one with the highest value is preferred. The dominance rule based on cumulative prospect theory can be formulated as follows:

CPT: Alternative L1 dominates alternative L2 (written as $\mathrm{L1} \succ_{\mathrm{CPT}} \mathrm{L2}$) if and only if $\mathrm{CPT}(\mathrm{L1}) > \mathrm{CPT}(\mathrm{L2})$.

3 Almost stochastic dominance rules

For years the most common decision rule under risk was the mean-variance (MV) rule proposed by Markowitz [10]. For risky alternative $\mathrm{L1}$ and $\mathrm{L2}$ with expected values $\mathrm{E(L1)}$, $\mathrm{E(L2)}$ and standard deviations $\sigma(\mathrm{L1})$, $\sigma(\mathrm{L2})$ the MV rule is following:

MV: Alternative L1 dominates alternative L2 (written as $\mathrm{L1} \succ_{\mathrm{MV}} \mathrm{L2}$) if and only if $\mathrm{E(L1)} \geq \mathrm{E(L2)}$ and $\sigma(\mathrm{L1}) \leq \sigma(\mathrm{L2})$ with at least one strict inequality.

Common accepted and objective nonparametric decision rule is the stochastic dominance. Lets $\mathrm{F_{L1}}$ and $\mathrm{F_{L2}}$ be the distribution functions of risky alternative $\mathrm{L1}$ and $\mathrm{L2}$ respectively, and S be a set of all outcomes of $\mathrm{L1}$ and $\mathrm{L2}$. The first and the second stochastic dominance rules are formulated as follows [4]:

FSD: Alternative $\mathrm{L1}$ dominates alternative $\mathrm{L2}$ by the first stochastic dominance (written as $\mathrm{L1} \succ_{\mathrm{FSD}} \mathrm{L2}$) if and only if inequality $\mathrm{F_{L1}}(r) - \mathrm{F_{L2}}(r) \leq 0$ is satisfied for each $r \in \mathrm{S}$ and for at least one value $r \in \mathrm{S}$ this inequality is strict.

SSD: Alternative $\mathrm{L1}$ dominates alternative $\mathrm{L2}$ by the second stochastic dominance (written as $\mathrm{L1} \succ_{\mathrm{SSD}} \mathrm{L2}$) if and only if inequality $\mathrm{F_{L1}^{(2)}}(r) - \mathrm{F_{L2}^{(2)}}(r) \leq 0$ is satisfied for each $r \in \mathrm{S}$ and for at least one value $r \in \mathrm{S}$ this inequality is strict, where $\mathrm{F_{L1}^{(2)}}(r) = \int_{-\infty}^{r} \mathrm{F_{L1}}(t)\, dt$ and $\mathrm{F_{L2}^{(2)}}(r) = \int_{-\infty}^{r} \mathrm{F_{L2}}(t)\, dt$.[3]

The MV and stochastic dominance rules often do not lead to the conclusion which alternative is better. In such situation we need other criteria for decision-making. Such situation is presented in example 1.

Example 1. The possible results of the risky alternative $\mathrm{L1}$ are to gain \$1 with probability 0.01 or to gain \$100 with probability 0.99, and in the alternative $\mathrm{L2}$ one can gain certain \$2. Both alternatives can be written as $\mathrm{L1} = ((1, 0.01); (100, 0.99))$ and $\mathrm{L2} = ((2, 1))$. It is easy to show that neither $\mathrm{L1}$ dominates $\mathrm{L2}$ nor $\mathrm{L2}$ dominates $\mathrm{L1}$ based on the MV rule. Also neither $\mathrm{L1}$ nor $\mathrm{L2}$ dominates the other based on the first or the second stochastic dominance rules, but most “reasonable” decision-makers (if not all) prefer $\mathrm{L1}$ to $\mathrm{L2}$. Moreover, analyzing graphs of both distribution functions showed in figure 3, we can notice that the area A corresponding to the range in which $\mathrm{L2}$ dominates $\mathrm{L1}$, is much smaller than the area B corresponding to the range in which $\mathrm{L1}$ dominates $\mathrm{L2}$. Therefore we can say that $\mathrm{L1}$ “almost” dominates $\mathrm{L2}$ by the first stochastic dominance.

[3] For discrete probability distributions the values $\mathrm{F_{L1}^{(2)}}(r)$ and $\mathrm{F_{L2}^{(2)}}(r)$ are cumulated values of the distribution functions (sums of the cumulated probabilities).

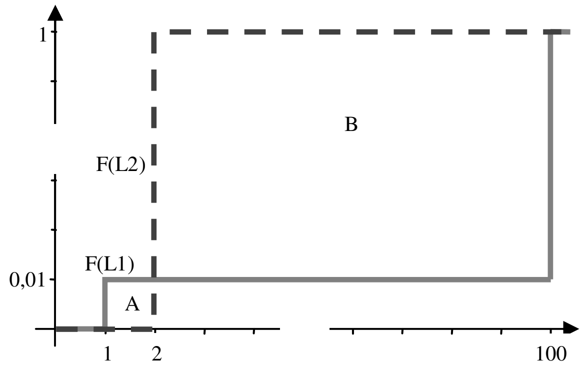

Figure 3 The distribution functions for L1 and L2

Analyzing similar examples Leshno and Levy proposed the concept of almost stochastic dominance (ASD) which is some relaxation of stochastic dominance rule [6]. The definitions of almost first and second stochastic dominance are as follows:

AFSD: Alternative L1 dominates alternative L2 by almost first stochastic dominance (written as $\mathrm{L1} \succ_{\mathrm{AFSD}} \mathrm{L2}$) if and only if

$$\int_{S_1} (F_{L1}(r) - F_{L2}(r))\, dr \le \varepsilon \int_{S} | F_{L1}(r) - F_{L2}(r) |\, dr$$

where S is a set of all outcomes of L1 and L2 and $S_1 = \{r \in S : F_{L2}(r) < F_{L1}(r)\}$.

ASSD: Alternative L1 dominates alternative L2 by almost second stochastic dominance (written as $\mathrm{L1} \succ_{\mathrm{ASSD}} \mathrm{L2}$) if and only if

$$\int_{S_2} (F_{L1}(r) - F_{L2}(r))\, dr \le \varepsilon \int_{S} | F_{L1}(r) - F_{L2}(r) |\, dr$$

and

$$\mathrm{E(L1)} \ge \mathrm{E(L2)}$$

where S is a set of all outcomes of L1 and L2 and $S_2 = \{r \in S_1 : F^{(2)}_{L2}(r) < F^{(2)}_{L1}(r)\}$.

It is assumed that the value of ε parameter connected with “actual” violation area should be less than 0.5 for both the first and the second stochastic dominance rules. [4]

In the example 1 neither alternative L1 nor alternative L2 dominates the other, and second stochastic dominance, but L1 dominates L2 by AFSD for $\varepsilon \approx 0.000103$ (parameter ε is defined as the area A divided by the total absolute area enclosed between both distribution functions (area A+B)).The main advantage of applying almost stochastic dominance rules is the possibility for reduction of a set of non-comparable (according to other criteria) risky alternatives. Moreover, almost stochastic dominance rules reveal preferences consistent with intuition, whereas traditional stochastic dominance rules may not confirm intuitional choices.

4 Consistency between preferences based on the cumulative prospect theory and the almost stochastic dominance rules

To analyze the consistency between preferences determined on the basis of the cumulative prospect theory and the almost stochastic dominance rules we have examined some examples of pairs of decision alternatives.

For alternatives L1 and L2 (showed in example 1) the selection of dominating alternative on the basis of the cumulative prospect theory is consistent with the selection based on almost stochastic dominance rules (sum-

[4] In literature [8] there are also defined $\varepsilon^*-\mathrm{AFSD}$ and $\varepsilon^*-\mathrm{ASSD}$, where ε^* indicates “allowed” violation area, and $0 < \varepsilon < \varepsilon^* < 0.5$.

mary of calculated values is in table 1 and 2). The question arises whether this consistency will always be observed?

Example 2. Let's consider two risky alternative $L3=\left((30,\,0.4);(60,\,0.6)\right)$ and $L4=\left((40,\,0.4);(50,\,0.6)\right)$. Similarly as for the pair $L1$ and $L2$ none of the alternative dominates the other by the MV rule (see table 1 and 2). There is also no dominance by FSD rule (what can be seen in figure 4) and SSD rule. But selections made by CPT and AFSD (and consequently ASSD) rules do not coincide. According to CPT rule $L4$ is the dominating alternative and according to AFSD (and also ASSD) $L3$ is the dominating one (see table 1 and 2).

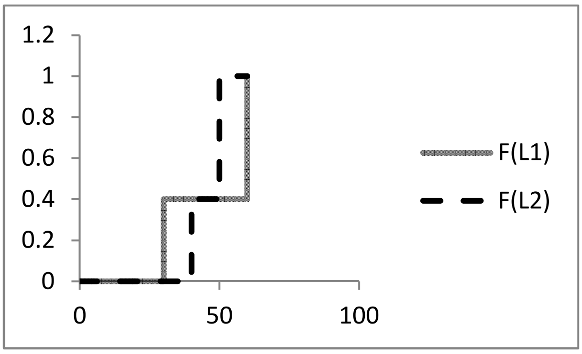

Figure 4 The distribution functions for $L3$ and $L4$

Example 3. Let's consider alternatives $L5=\left((20,0.2);(30,0.5);\,(56,0.3)\right)$ and $L6=\left((10,0.1);(28,0.5);\,(52,0.4)\right)$. The alternative $L5$ dominates $L6$ by MV rule, but there is no dominance by FSD, SSD and AFSD (see table 1 and 2). It is worth to notice that selections based on the CPT and ASSD do not coincide. Alternative $L5$ is dominating according to CPT rule and alternative $L6$ is dominating by ASSD. Our example is also interesting because both ε (for ASSD) are less than 0,5 and both expected values are the same. In such case the alternative with lower epsilon dominates [6]. Therefore $L6$ dominates $L5$ by ASSD rule.

	Alternatives					
Parameter	L1	L2	L3	L4	L5	L6
E(L)	99.01	2.00	48.00	46.00	35.80	35.80
σ(L)	9.85	0.00	14.70	4.90	13.75	14.21
CPT(L)	52.54	1.84	27.89	28.33	22.24	20.58
ε_{AFSD}	0.000103 for $(L1, L2)$		0.4 for $(L3, L4)$		0.5 for $(L5, L6)$	
	0.999897 for $(L2, L1)$		0.6 for $(L4, L3)$		0.5 for $(L6, L5)$	
ε_{ASSD}	0.000103 for $(L1, L2)$		0.4 for $(L3, L4)$		0.366667 for $(L5, L6)$	
	0.999897 for $(L2, L1)$		0.6 for $(L4, L3)$		0.3 for $(L6, L5)$	

Table 1 Values of parameters for alternatives $L1-L6$

Decision rule	Example 1	Example 2	Example 3
MV	-	-	$L5 \succ L6$
CPT	$L1 \succ L2$	$L4 \succ L3$	$L5 \succ L6$
FSD	-	-	-
SSD	-	-	-
AFSD	$L1 \succ L2$	$L3 \succ L4$	-
ASSD	$L1 \succ L2$	$L3 \succ L4$	$L6 \succ L5$

Table 2 Dominances for alternatives $L1-L6$

Mutual dependences between considered decision rules are showed on diagram in figure 5.

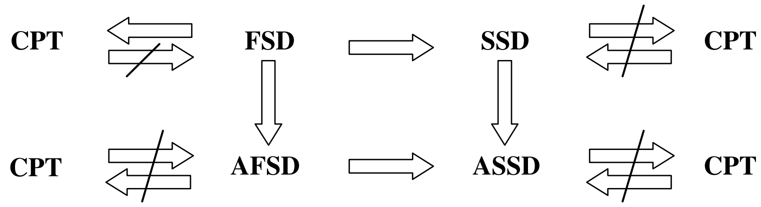

Figure 5 Dependences between FSD, SSD, AFSD, ASSD and CPT rules

The dependences showed in figure 5 between FSD, SSD, AFSD and ASSD dominances are supported by the literature [6, 7]. In the analyzed examples we showed that there is no consistency between choices based on cumulative prospect theory and the almost first and second stochastic dominance. Correctness of implication FSD $\Rightarrow$ CPT is confirmed in [9, 13], whereas converse implication do not occur [11]. Some examples showing no consistency between CPT and SSD rules are presented in [11].

5 Summary

The motivation of the authors of the cumulative prospect theory as well as the almost stochastic dominance rules was to create good tools for modeling real choices. However, as we showed in our article choices made on the basis of considered tools are not always consistent. The identification of causes of this discrepancy will need further research.

References

[1] Currim I., Sarin R.: Prospect versus utility. *Management Science* **35** (1989), No. 1, 22–41.
[2] Dudzińska-Baryła R., Kopańska-Bródka D.: Maximum expected utility portfolios versus prospect theory approach. In: *Increasing competitiveness or regional, national and international markets development..Proceedings of the 25th International Conference on Mathematical Methods in Economics 2007.* Technical University of Ostrava, Ostrava, 2007, electronic document.
[3] Gonzalez R., Wu G.: On the shape of the probability weighting function. *Cognitive Psychology* **38** (1999), 129–166.
[4] Hanoch G., Levy H.: The efficiency analysis of choices involving risk. *Review of Economic Studies* **36** (1969), No. 3, p. 335–346.
[5] Kahneman D., Tversky A.: Prospect Theory: An Analysis of Decision Under Risk. *Econometrica* **47** (1979), 263–291.
[6] Leshno M., Levy H.: Preferred by "all" and preferred by "most" decision makers: Almost stochastic dominance. *Management Science* **48** (2002), Issue 8, 1074–1085.
[7] Levy H.: Stochastic dominance and expected utility: survey and analysis. *Management Science* **38** (1992), No. 4, 555–593.
[8] Levy H., Leshno M., Leibovitch B.: Economically relevant preferences for all observed epsilon. *Annals of Operations Research* **176** (2010), 153–178.
[9] Levy H., Wiener Z.: Prospect theory and utility theory: temporary versus permanent attitude towards risk. *http://pluto.mscc.huji.ac.il/~mswiener/research/PTandUT.pdf* , 2002
[10] Markowitz H.: Portfolio selection. *The Journal of Finance* **7** (1952), No. 1, 77–91.
[11] Michalska E.: Dominacje stochastyczne a teoria perspektyw. In: *Współczesne tendencje rozwojowe badań operacyjnych,* prace naukowe nr 108 (Siedlecki, J., Peternek, P., eds.), Wydawnictwo Uniwersytetu Ekonomicznego we Wrocławiu, Wrocław, 2010, 167–176.
[12] Prelec D.: The probability weighting function. *Econometrica* **66** (1998), No. 3, 497–527.
[13] Tversky A., Kahneman D.: Advances in prospect theory: cumulative representation of uncertainty. *Journal of Risk and Uncertainty* **5** (1992), 297–323.
[14] Wu G., Gonzalez R.: Curvature of the probability weighting function. *Management Science* **42** (1996), 1676–1690.